

**Problem 3.3.26 Solution**

(a) For the matrix C, the reduced row-echo form of C is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

And we can also find rref of H, X,Y, T, L. and  $rref(C) = rref(L)$ we know that  $KerC = Ker(rref(C))$ , therefore matrix L has the same kernel with C.

(b): For the matrix C,  $C^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ , then  $rref(C^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore,  $ImC = Im(rref(C^T)^T)$

By the same way, we can get the reduced row-echo form of  $H^T, L^T, T^T, X^T, Y^T$ . And we find that  $H^T$  and  $X^T$  has the same reduced row-echo form with matrix  $C^T$ , therefore, C, H, X has the same image.

(c) By the same way as (b), we know the answer is matrix L.

**Problem 3.3.28 Solution** The problem means to decide k to make the ma-

trix:  $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & k \end{bmatrix}$  as an invertible matrix. OK, reduce the matrix to an upper

triangle matrix, which is  $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k - 29 \end{bmatrix}$

Therefore, the answer is  $k \neq 29$

**Problem 3.3.36 Solution** The answer is NO. because  $dim(Im(A))+dim(ker(A)) = 3$ .  $ImA = KerA$ , means  $dim(Im(A)) = 1.5$  which is impossible.

**Problem 3.3.44 solution**

Proof:  $\Rightarrow$  Proof by contradiction. Firstly, suppose both this two subspaces are not  $R^n$ , because if one of them is  $R^n$ , then it is trivial case. Now, for any vector  $\vec{x}$ , we can find unique  $\vec{v} \in V, \vec{w} \in W$ , such that  $\vec{x} = \vec{v} + \vec{w}$ , if  $V \cap W \neq 0$ , then we can find a vector  $\vec{y} \in V \cap W$ , and

$\vec{y} = \vec{0} + \vec{y}$ , while  $\vec{0} \in V$  and  $\vec{y} \in W$ . Also  $\vec{y} = \vec{y} + \vec{0}$ , while  $\vec{y} \in V$  and  $\vec{0} \in W$ . Contradiction to the uniqueness condition. therefore  $V \cap W = \{\vec{0}\}$ .

About dimension, suppose  $dimV = s$  and  $dimW = t$ , then we can find basis  $\{\vec{a}_1, \dots, \vec{a}_s\}$  of V, and basis  $\{\vec{b}_1, \dots, \vec{b}_t\}$  of W. We claim, they are linear inde-

pendent, so they form the basis of  $R^n$ . If not, then suppose  $b_i = y_1 \vec{a}_1 + \dots + y_s \vec{a}_s$  contradiction to  $V \cap W = \{\vec{0}\}$ .

$\Leftarrow$  Proof by contradiction, if not unique, that means  $\exists \vec{x}$ , such that  $\vec{x} = \vec{v}_1 + \vec{w}_1$  and  $\vec{x} = \vec{v}_2 + \vec{w}_2$  while  $\vec{v}_1, \vec{v}_2$  are in  $V$ ,  $\vec{w}_1, \vec{w}_2$  are in  $W$ .

$$\text{So } \vec{x} = \vec{v}_1 + \vec{w}_1 = \vec{v}_2 + \vec{w}_2$$

Which means  $\vec{v}_1 - \vec{v}_2 = \vec{w}_2 - \vec{w}_1$ . and  $\vec{v}_1 - \vec{v}_2 \in V$ ,  $\vec{w}_2 - \vec{w}_1 \in W$ .

Because  $V \cap W = \{\vec{0}\}$ , so  $\vec{v}_1 - \vec{v}_2 = 0$  and  $\vec{w}_2 - \vec{w}_1 = 0$ . that means  $\vec{v}_1 = \vec{v}_2$ , and  $\vec{w}_1 = \vec{w}_2$  contradiction.

End of proof.