

### 3.4.16 Solution

We do it by Gauss-Jordan:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \end{array} \right] \begin{array}{l} \\ -(I) \\ -(I) \end{array}$$

then:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & -6 \\ 0 & 2 & 5 & -4 \end{array} \right] \begin{array}{l} \\ -(II) \\ -2 \times (II) \end{array}$$

then:

$$\left[ \begin{array}{ccc|c} 1 & 0 & -1 & 13 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & 8 \end{array} \right] \begin{array}{l} +(III) \\ \\ -2 \times (III) \end{array}$$

then:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 8 \end{array} \right]$$

Therefore,  $\vec{x}$  is in  $V$  and  $[\vec{x}]_B = \begin{bmatrix} 21 \\ -22 \\ 8 \end{bmatrix}$

### 3.4.30 Solution

So

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

then:

$$S^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

We have  $B = S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**3.4.46 Solution**

Firstly, choose any vector  $\vec{v}$  in the plane  $x_1 + 2x_2 + x_3 = 0$ . then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, we have a basis for the plane which is  $\{v_1, v_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Suppose the basis we want is  $\{w_1, w_2\}$ .

then assume  $w_1 = av_1 + bv_2$ ,  $w_2 = cv_1 + dv_2$ . Because they are basis, which means they are linear independent. therefore:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

That means  $ad - bc \neq 0$

And we know that  $[v]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

So,  $2w_1 - w_2 = \begin{bmatrix} 2c - 4a + d - 2b \\ 2a - c \\ 2b - d \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

So we have:  $2a - c = -1$  and  $2b - d = 1$ , which are equivalent to  $c = 1 + 2a$ ,  $d = 2b - 1$

And we know  $ad - bc \neq 0$ . We plug  $c$  and  $d$  in, then we have:  $2ab - a - 2ab - b \neq 0$  which means  $a + b \neq 0$

Now we choose any a,b satisfying  $a + b \neq 0$ , for example let  $a = 1$ ,  $b = 1$ , then  $c = 3$ ,  $d = 1$ . The new basis is:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix} \right\}$$

BTW: there are many choices, so the answer is not unique.

**4.1.8 Solution** The answer is YES.