Homework 6 Solutions

April 17, 2005

pp. 162-163

(50) Just write out the characteristic equation, which is

$$\lambda^2 + 8\lambda - 20 = (\lambda + 10)(\lambda - 2) = 0$$

.
Hence two linearly independent solutions are

$$f_1(x) = e^{-10x} \quad f_2(x) = e^{2x}$$

Since the space of solutions to a linear differential equation of second order is two-dimensional, the two functions above are a basis for this space, and so any solution of the equation can be written in the form $g(x) = Af_1 + Bf_2$ for some $A, B \in \mathbb{R}$.

(58) Part (a): We find

$$\frac{d}{dx}(g(x)^2 + g'(x)^2) = 2g'(x)(g(x) + g''(x)) = 0$$

if $g(x) \in V$. Thus the function $g^2 + (g')^2$ is constant.
Part (b): Using part (a), we know that the equation

$$h(x) = g(x)^2 + g'(x)^2 = a$$

for some constant a. Hence $h(x) = h(0)$ for all x. But $h(0) = 0$ from the assumption $g(0) = g'(0) = 0$. Thus $h(x) = 0$ everywhere, and it follows that $g = 0$ everywhere.

Part (c): Note that both $\cos(x)$ and $\sin(x)$ are in V. This can be proved by straightforward differentiation. Since V is a linear space, it follows that if $f(x)$ is in V, so is

$$g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x).$$

Clearly $g(0) = 0$. Differentiating g, we find

$$g'(0) = f'(0) - f(0)\sin(0) - f'(0)\cos(0) = 0$$

and by part (b) this implies that $g = 0$ identically. We are finished.

pp. 169 - 171

(70) To say that $p(x) \in \mathrm{Ker}(T)$ means that $p(c_i) = 0$ for $i = 0, \dots n$. If the c_i are all distinct, this is not possible, for then $p(x)$ would be divisible by

$$h(x) = \prod_{i=0}^{n} (x - c_i) \in P_{n+1}$$

an impossibility since $p \in P_n$. Thus if the c_i are all distinct, T has a trivial kernel, and so is an isomorphism because P_n and $\mathbb{R}^{n+1}$ have the same dimension. If the c_i are not all distinct, then the following polynomial

$$h(x) = \prod_{\text{distinct } c_i} (x - c_i)$$

lies in P_n and vanishes at all the c_i; thus $T(h) = 0$ and T fails to be an isomorphism. Hence T is an isomorphism iff all the c_i are distinct.

(76) Note

$$T(0_V) = T(0_V + 0_V) = 2T(0_V)$$

using linearity of T. Hence $T(0_V) = 2T(0_V)$. This is only possible if $T(0_V) = 0_W$.