Homework 7 Solutions

April 17, 2005

pp. 208-209

(14) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix}$$

First, normalize them all:

$$\hat{v}_1 = \begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix}, \quad \hat{v}_2 = \begin{bmatrix} 0 \\ 7/10 \\ 1/5 \\ 7/10 \end{bmatrix}, \quad \hat{v}_3 = \begin{bmatrix} 1/\sqrt{102} \\ 8/\sqrt{102} \\ 1/\sqrt{102} \\ 6/\sqrt{102} \end{bmatrix}$$

To find an orthonormal basis $\hat{w}_i,\ i = 1, 2, 3$, let $\hat{w}_1 = \hat{v}_1$, and

$$\hat{w}_2 = \frac{\hat{v}_2 - (\hat{v}_1 \cdot \hat{v}_2)\hat{v}_1}{\|\hat{v}_2 - (\hat{v}_1 \cdot \hat{v}_2)\hat{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Finally,

$$\hat{w}_3 = \frac{\hat{v}_3 - (\hat{w}_1 \cdot \hat{v}_3)\hat{w}_1 - (\hat{w}_2 \cdot \hat{v}_3)\hat{w}_2}{\|\hat{v}_3 - (\hat{w}_1 \cdot \hat{v}_3))\hat{w}_1 - (\hat{w}_2 \cdot \hat{v}_3)\hat{w}_2\|}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

At the end, do a reality check: yes, each of the w_i is orthogonal to the others, and yes, they all have unit length. We are done (whew!)

(28) Read off the answer from the steps in (14) and Fact 5.2.2 :

$$Q = \begin{pmatrix} 1/10 & -1/\sqrt{2} & 0 \\ 7/10 & 0 & 1/\sqrt{2} \\ 1/10 & 1/\sqrt{2} & 0 \\ 7/10 & 0 & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 \end{pmatrix}$$

$$R = \begin{pmatrix} 10 & \hat{w}_1 \cdot \vec{v}_2 & \hat{w}_1 \cdot \vec{v}_3 \\ 0 & \|\vec{v}_2^{\perp}\| & \hat{w}_2 \cdot \vec{v}_3 \\ 0 & 0 & \|\vec{v}_3^{\perp}\| \end{pmatrix} = \begin{pmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

Reality check: multiply the matrices; yes, it works out.

(34) After getting A into rref we find that

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

are a basis for $\ker(A)$. Just perform Graham-Schmidt to give an orthonormal basis:

$$\hat{w}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{w}_2 = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ -1 \\ -4 \\ 1 \end{pmatrix}$$

(40) Use Fact 5.2.2. If $\vec{v}_i, i = 1, \ldots k$ are the columns of A, the QR-factorization looks like

$$A = \begin{pmatrix} \hat{v}_1 & \cdots & \hat{v}_k \end{pmatrix} \times \operatorname{diag}(\|\vec{v}_1\|, \ldots, \|\vec{v}_k\|)$$

where $\operatorname{diag}(a_1, \ldots a_k)$ means the $k \times k$ diagonal matrix with entries a_i, and the $\hat{v}_i$ are the unit vectors in the directions of the $\vec{v}_i$.