Linear Algebra
HW8
p246 40, 44, 56, 60

$\boxed{5.3.40}$ $W \subset \mathbb{R}^4$. Spanned by $\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix} \right\}$

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{pmatrix} -1 \\ 7 \\ -7 \\ 1 \end{pmatrix}$

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2\|} = \frac{1}{10}\begin{pmatrix} -1 \\ 7 \\ -7 \\ 1 \end{pmatrix}$.

$$QQ^T = \begin{pmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/10 & 7/10 & -7/10 & 1/10 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 10 & 30 & -5 & 15 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{pmatrix}$$

$\boxed{5.3.44}$ $A_{n \times m}$. $\dim(\text{im}(A)) + \dim(\ker(A^T))$
$= \text{rank } A + \text{nullity } A^T$
$= \text{rank } A^T + \text{nullity } A^T = n$, since $A^T_{m \times n}$.

$\boxed{5.3.56}$ $L(A) = A^T : \mathbb{R}^{2\times3} \to \mathbb{R}^{3\times2}$: $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \longmapsto \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$.

we know $\mathbb{R}^{2\times3} \cong \mathbb{R}^{3\times2} \cong \mathbb{R}^6$. So choose a basis

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\mathbb{R}^{3\times2}$, and a basis $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

of $\mathbb{R}^{2\times3}$. Then, under the natural mappings $\mathbb{R}^{2\times3} \to \mathbb{R}^6 \to \mathbb{R}^{3\times2}$,

$$\begin{tikzcd} \mathbb{R}^{2\times3} \arrow[r] \arrow[rr, bend right, "M"'] & \mathbb{R}^6 \arrow[r] & \mathbb{R}^{3\times2} \end{tikzcd}$$

we have that $M = [T(\vec{v}_1)_B \cdots T(\vec{v}_6)_B]$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I_6.$$

$\boxed{5.3.60}$ $L(A) = A^T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$,

with $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$$M = [T(\vec{v}_1)_B \cdots T(\vec{v}_4)_B] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$