(2) The two vectors making up the columns of $A$ are linearly independent, and so they form a basis for the image of $A$. The fundamental theorem $(\text{im}(A))^\perp = \ker(A^T)$ tells us that a basis of $\ker(A^T)$ can be found by finding a nonzero vector perpendicular to the plane spanned by the columns of $A$. Such a vector, by any number of methods, is given by

$$\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$ 

(10) (a) Use the fact that each $\vec{x}$ has a unique representation

$$\vec{x} = \vec{x}_0 + \vec{x}_1$$

where $\vec{x}_0 \in (\ker(A))^\perp$ and $\vec{x}_1 \in \ker(A)$. If $A\vec{x} = \vec{b}$, then we have

$$A\vec{x} = A(\vec{x}_0 + \vec{x}_1) = A\vec{x}_0 = \vec{b}$$

proving the claim. The basic point here is that the kernel of $A$ does not contribute when solving inhomogeneous linear equations.

(b) Suppose that $\vec{x}_1$ and $\vec{x}_2$ are two solutions to the system. Then clearly $\vec{x}_1 - \vec{x}_2$ is in the kernel of $A$. But if both $\vec{x}_1$ and $\vec{x}_2$ are perpendicular to $\ker(A)$, then so is their difference. The only vector in both $\ker(A) \cap (\ker(A))^\perp$ is the zero vector, and this finishes the demonstration.

(c) By (b) any other solution $\vec{x}$ to the system is of the form $\vec{x} = \vec{x}_0 + \vec{c}$, where $\vec{c} \in \ker(A)$. By the Pythagorean theorem,
\[ \| \vec{x} \|^2 = \| \vec{x}_0 \|^2 + \| \vec{c} \|^2 \]
from which the claim follows immediately.

(16) Let \( A : \mathbb{R}^m \to \mathbb{R}^n \). From the FTLA, we have the relation
\[ \text{rk}(A^T) + \text{null}(A^T) = n \]
But from the relation \((\text{im}(A))^\perp = \ker(A^T)\), we see immediately that
\[ \text{null}(A^T) = n - \dim(\text{im}(A)) = n - \text{rk}(A) \]
. Thus
\[ \text{null}(A^T) = n - \text{rk}(A^T) = n - \text{rk}(A), \]
so that \( \text{rk}(A) = \text{rk}(A^T) \).

(20) All you have to do is find the projection \( \tilde{b}^* \) of \( \tilde{b} \) onto \( \text{im}(A) \) and solve the system \( A\vec{x}^* = \tilde{b}^* \). A normal vector perpendicular to the image of \( A \) is
(just find the equation of the plane defining \( \text{im}(A) \))
\[ \hat{v} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix} \]
By straightforward computation, then
\[ \tilde{b}^* = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} \]
One easily verifies that
\[ A \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \tilde{b}^*, \]
so that \( \vec{x}^* = [2 \ 2] \).