Homework 8 Solutions

April 17, 2005

pp. 228-232

(2) The two vectors making up the columns of A are linearly independent, and so they form a basis for the image of A. The fundamental theorem $(\mathrm{im}(A))^\perp = \ker(A^T)$ tells us that a basis of $\ker(A^T)$ can be found by finding a nonzero vector perpendicular to the plane spanned by the columns of A. Such a vector, by any number of methods, is given by

$$\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

(10) (a) Use the fact that each $\vec{x}$ has a unique representation

$$\vec{x} = \vec{x}_0 + \vec{x}_1$$

where $\vec{x}_0 \in (\ker(A))^\perp$ and $\vec{x}_1 \in \ker(A)$. If $A\vec{x} = \vec{b}$, then we have

$$A\vec{x} = A(\vec{x}_0 + \vec{x}_1) = A\vec{x}_0 = \vec{b}$$

proving the claim. The basic point here is that the kernel of A does not contribute when solving inhomogeneous linear equations.

(b) Suppose that $\vec{x}_1$ and $\vec{x}_2$ are two solutions to the system. Then clearly $\vec{x}_1 - \vec{x}_2$ is in the kernel of A. But if both $\vec{x}_1$ and $\vec{x}_2$ are perpendicular to $\ker(A)$, then so is their difference. The only vector in both $\ker(A) \cap (\ker(A))^\perp$ is the zero vector, and this finishes the demonstration.

(c) By (b) any other solution $\vec{x}$ to the system is of the form $\vec{x} = \vec{x}_0 + \vec{c}$, where $\vec{c} \in \ker(A)$. By the Pythagorean theorem,

$$\|\vec{x}\|^2 = \|\vec{x}_0\|^2 + \|\vec{c}\|^2$$

from which the claim follows immediately.

(16) Let $A : \mathbb{R}^m \to \mathbb{R}^n$. From the FTLA, we have the relation

$$\mathrm{rk}(A^T) + \mathrm{null}(A^T) = n$$

But from the relation $(\mathrm{im}(A))^\perp = \ker(A^T)$, we see immediately that

$$\mathrm{null}(A^T) = n - \dim(\mathrm{im}(A)) = n - \mathrm{rk}(A)$$

. Thus

$$\mathrm{null}(A^T) = n - \mathrm{rk}(A^T) = n - \mathrm{rk}(A),$$

so that $\mathrm{rk}(A) = \mathrm{rk}(A^T)$.

(20) All you have to do is find the projection $\vec{b}^*$ of $\vec{b}$ onto $\mathrm{im}(A)$ and solve the system $A\vec{x}^* = \vec{b}^*$. A normal vector perpendicular to the image of A is (just find the equation of the plane defining $\mathrm{im}(A)$)

$$\hat{v} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

By straightforward computation, then

$$\vec{b}^* = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}.$$

One easily verifies that

$$A \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \vec{b}^*,$$

so that $\vec{x}^* = [2\ 2]$.