

Problem 5.5.8

Yes it is a linear operation, and its image is all the real numbers, which is \mathbb{R} . its kernel is a subspace of V which is perpendicular to w .

Problem 5.5.10

Let $g(t) = a + bt + ct^2$ in the space of all functions in P_2 that are orthogonal to $f(t) = t$, then

$$\frac{1}{2} \int_{-1}^1 at + bt^2 + ct^3 dt = 0$$

then

$$\frac{1}{4}at^2 + \frac{1}{6}bt^3 + \frac{1}{8}ct^4 \Big|_{-1}^1 = 0$$

So $b = 0$ Now we have $g(t) = a + ct^2$ basis for the space is $\{v_1 = 1, v_2 = t^2\}$, now we use Gram-Schmidt to find the orthonormal basis.

$$\|1\| = \sqrt{\frac{1}{2} \int_{-1}^1 1 dt} = \sqrt{\frac{1}{2}t \Big|_{-1}^1} = 1$$

So

$$u_1 = \frac{1}{\|1\|} = 1$$

$$v_2^\perp = v_2 - (v_2, u_1)u_1 = t^2 - \frac{1}{2} \int_{-1}^1 t^2 dt = t^2 - \frac{1}{6}t^3 \Big|_{-1}^1 = t^2 - \frac{1}{3}$$

then,

$$\|v_2^\perp\| = \sqrt{\frac{1}{2} \int_{-1}^1 (t^2 - \frac{1}{3})^2 dt} = \sqrt{\frac{1}{2} \int_{-1}^1 t^4 - \frac{2}{3}t^2 + \frac{1}{9} dt} = \sqrt{\frac{1}{10}t^5 - \frac{1}{9}t^3 + \frac{1}{18}t \Big|_{-1}^1} = \sqrt{\frac{4}{45}} = \frac{2}{3\sqrt{5}}$$

Therefore,

$$u_2 = \frac{v_2^\perp}{\|v_2^\perp\|} = \frac{3\sqrt{5}}{2} (t^2 - \frac{1}{3})$$

5.5.14

check by definition of inner product

(a) No, the counter example is $f(t) = (t-1)(t-2)$ then $(f, f) = 0$

(b) Yes, directly check. here we know that a polynomial in P_2 can only have two roots, and actually for (a), "(,)" satisfies all the conditions except the last one $(f, f) > 0$ for all nonzero f . And here, we have no way to make a nonzero f in P_2 to have three roots, then " \ll, \gg " is an inner product.

Problem 5.5.16

(a) So the standard basis for P_1 is $\{v_1 = 1, v_2 = t\}$, now we use Gram-Schmidt to make it orthonormal,

$$\|1\| = \sqrt{\int_0^1 1 dt} = \sqrt{t} \Big|_0^1 = 1$$

therefore

$$u_1 = 1$$

Then,

$$v_2^\perp = v_2 - (v_2, u_1)u_1 = t - \int_0^1 t dt = t - \frac{1}{2}$$

and,

$$\|v_2^\perp\| = \sqrt{\int_0^1 (t - \frac{1}{2})^2 dt} = \sqrt{\int_0^1 t^2 - t + \frac{1}{4} dt} = \sqrt{\frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{4}t} \Big|_0^1 = \frac{1}{\sqrt{12}}$$

So

$$u_2 = \frac{v_2^\perp}{\|v_2^\perp\|} = \sqrt{12}(t - \frac{1}{2})$$

(b)

$$Proj_{P_1} f = (1, f) + 12(t - \frac{1}{2}, f)(t - \frac{1}{2}) = \int_0^1 t^2 dt + 12(t - \frac{1}{2}) \int_0^1 t^3 - \frac{1}{2}t^2 dt = t - \frac{1}{6}$$