7.3 16. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & -1-\lambda & -1 \\ 2 & 2 & -\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(-1-\lambda)(-\lambda) + 2(1-\lambda) - 2 \\ &= (\lambda^2 - 1)(-\lambda) - 2\lambda \\ &= -\lambda(\lambda^2 + 1) \end{aligned}$$

Only real eigenvalue is 0.

Its eigenspace is $\ker A$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x - z = 0 \qquad \begin{pmatrix} t \\ -t \\ t \end{pmatrix}$$
$$y + z = 0$$

$$\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

20. Eigenvalues are 1 and 2. 1 has alg mult 2.

$$A - I = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$ay = 0$$
$$z = 0$$

If $a = 0$, then $\ker A - I = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$ geo mult is 2

If $a \neq 0$, then $\ker A - I = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$ geo mult is 1

Eigenvalue 2 has geo mult 1 always.

Eigenbasis exists when $a = 0$.

$\boxed{7.4}$ 16.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)^3 - (1-\lambda) \\ &= (1-\lambda)\left[(1-\lambda)^2 - 1\right] \\ &= (1-\lambda)\ (\lambda^2 - 2\lambda) \\ &= (1-\lambda)(\lambda - 2)\ \lambda \end{aligned}$$

eigenvalues are $0, 1, 2$

0-eigenspace is ker A : $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{aligned} x + z &= 0 \\ y &= 0 \end{aligned} \longrightarrow \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix}$$

$\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

1-eigenspace is ker A-I : $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{aligned} x &= 0 \\ z &= 0 \end{aligned} \longrightarrow \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$$

$\ker(A - I) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

2-eigenspace is ker A-2I : $\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{aligned} x - z &= 0 \\ y - 2z &= 0 \end{aligned} \longrightarrow \begin{pmatrix} t \\ 2t \\ t \end{pmatrix}$$

$\ker(A - 2I) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$

eigenbasis : $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

So we can use $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

54. $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\qquad$ $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\qquad$ $B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

So A is not similar to B, because if it were then $B = S^{-1}AS$

so then $B^2 = (S^{-1}AS)(S^{-1}AS) = S^{-1}A^2S = 0$

But this is not the case.

$\boxed{7.5}$ 24.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 5 & -7 & 3-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2(3-\lambda) - 7\lambda + 5 \\ &= -\lambda^3 + 3\lambda^2 - 7\lambda + 5 \\ &= (1-\lambda)(\lambda^2 - 2\lambda + 5) \end{aligned}$$

1 is a root

quadratic formula: roots are $\frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

eigenvalues are 1, $1+2i$, $1-2i$

27. A has 2 distinct eigenvalues.
One has alg multiplicity 1, call it λ_1.
The other has alg multiplicity 2, call it λ_2.
tr A is the sum of eigenvalues : $\lambda_1 + \lambda_2 + \lambda_2$
det A is the product : $\lambda_1 \lambda_2^2$
So $\lambda_1 + 2\lambda_2 = 1$ and $\lambda_1 \lambda_2^2 = 3$
$\lambda_1 = (1 - 2\lambda_2)$ so $(1-2\lambda_2)\lambda_2^2 = 3$
that is $-2\lambda_2^3 + \lambda_2^2 = 3$
i.e. $2\lambda_2^3 - \lambda_2^2 + 3 = 0$
So λ_2 is a root of $2x^3 - x^2 + 3 = (x+1)(2x^2 - 3x + 3)$

-1 is a root.

quadratic formula: $\frac{3 \pm \sqrt{9-24}}{4}$: Not Real

So the only real root of $2x^3 - x^2 + 3$ is -1
If we show λ_2 is real, then $\lambda_2 = -1$
If λ_2 is not real, then $\overline{\lambda_2}$ is a distinct eigenvalue.
So $\lambda_1 = \overline{\lambda_2}$
But λ_2 and $\overline{\lambda_2}$ would have to have
the same algebraic multiplicity.
This would be a contradiction,
so λ_2 is real, so $\lambda_2 = -1$.
Going back to the equation $\lambda_1 + 2\lambda_2 = 1$
this becomes $\lambda_1 - 2 = 1$, i.e. $\lambda_1 = 3$
Eigenvalues are 3 (multiplicity 1)
and -1 (multiplicity 2)

$\boxed{8.1}$ 9.

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2(2-\lambda) - 9(2-\lambda) \\ &= (2-\lambda)(\lambda^2 - 9) \\ &= (2-\lambda)(\lambda-3)(\lambda+3) \end{aligned}$$

eigenvalues are $2, 3, -3$

$\lambda = 2$: $A - 2I = \begin{pmatrix} -2 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{matrix} x = 0 \\ z = 0 \end{matrix} \longrightarrow \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$

$\ker(A - 2I) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

$\lambda = 3$: $A - 3I = \begin{pmatrix} -3 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{matrix} x - z = 0 \\ y = 0 \end{matrix} \longrightarrow \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$

$\ker(A - 3I) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

$\lambda = -3$: $A + 3I = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{matrix} x + z = 0 \\ y = 0 \end{matrix} \longrightarrow \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix}$

$\ker(A + 3I) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenbasis.

These eigenvectors are orthogonal.

So $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

is an orthonormal eigenbasis.

12. a. L is reflection across a line, ℓ.

Eigenvalues are 1 and -1

Vectors in ℓ are the eigenvectors for $\lambda = 1$

Vectors orthogonal to ℓ are the eigenvectors for $\lambda = -1$

$\ell = \text{span}\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)$

$\ell^\perp$: $x + 2z = 0$: $\ell^\perp = \text{span}\left(\begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

An eigenbasis is $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

This is already orthogonal, so

an orthonormal eigenbasis is $\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\2\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

b. $T\left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\2\end{bmatrix}\right) = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}$

$\left[T\left(\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\2\end{bmatrix}\right)\right]_B = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$T\left(\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\0\\1\end{bmatrix}\right) = -\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\0\\1\end{bmatrix}$

$\left[T\left(\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\0\\1\end{bmatrix}\right)\right]_B = \begin{pmatrix}0\\-1\\0\end{pmatrix}$

$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$

$\left[T\begin{pmatrix}0\\1\\0\end{pmatrix}\right]_B = \begin{pmatrix}0\\0\\-1\end{pmatrix}$

So B-matrix is $\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{pmatrix}$

$\boxed{8.2}$ 6. The associated matrix is $A = \begin{pmatrix}2 & 3\\3 & 4\end{pmatrix}$

$A - \lambda I = \begin{pmatrix}2-\lambda & 3\\3 & 4-\lambda\end{pmatrix}$

$\det(A - \lambda I) = (2-\lambda)(4-\lambda) - 9 = \lambda^2 - 6\lambda - 1$

roots are $\frac{6 \pm \sqrt{36+4}}{2} = 3 \pm \frac{1}{2}\sqrt{40} = 3 \pm \sqrt{10}$

$3 + \sqrt{10}$ is positive

$3 - \sqrt{10}$ is negative $\longrightarrow$ $(9 < 10$ so $3 < \sqrt{10})$

So q is indefinite.

We could also have seen this by computing $q(1,0)$ and $q(-3,2)$

$q(1,0) = 1 > 0$

$q(-3,2) = 18 - 36 + 8 = -10 < 0$

9. a. $(A^2)^T = A^T A^T = (-A)(-A) = (-1)^2 A = A$

So A is symmetric

and is not skew-symmetric (unless $A = 0$)

b. Let λ be an eigenvalue of A^2

with eigenvector v.

Notice $(Ax)\cdot y = (Ax)^T y = x^T A^T y = x^T (A^T y) = x \cdot A^T y$

$\lambda(v\cdot v) = (\lambda v)\cdot v = (A^2 v)\cdot v = Av \cdot A^T v$

$= Av\cdot(-Av) = -(Av\cdot Av) \leq 0$

So $\lambda(v\cdot v) \leq 0$ and $(v\cdot v) > 0$

This means $\lambda \leq 0$

So all eigenvalues of A^2 are ≤ 0

i.e. A^2 is negative semi-definite

c. Let λ be an eigenvalue of A

with eigenvector v.

Then $A^2 v = A(Av) = A(\lambda v) = \lambda Av = \lambda^2 v$

So λ^2 is an eigenvalue of A^2.

This means λ^2 is a non-positive real number by part b.

So λ is pure imaginary i.e.

$\lambda = ai$ for a real number a.

To be diagonalizable over $\mathbb{R}$, all the eigenvalues

must be real, so they are all zero.

Moreover, then A is similar to the zero matrix,

which means A is the zero matrix.

So the only skew-symmetric matrix which

is diagonalizable over $\mathbb{R}$ is the zero matrix.