Homework I solutions

1.1/16)

$$\begin{aligned} x+4y+z&=0\\ 4x+13y+7z&=0\\ 7x+22y+13z&=0\end{aligned} \xrightarrow[\mathrm{III}-7\mathrm{I}]{\mathrm{II}-4\mathrm{I}} \begin{aligned} x+4y+z&=0\\ -3y+3z&=0\\ -6y+6z&=0\end{aligned} \xrightarrow[\mathrm{III}\div(-6)]{\mathrm{II}\div(-3)} \begin{aligned} x+4y+z&=0\\ y-z&=0\\ y-z&=0\end{aligned}$$

$$\xrightarrow{\mathrm{III}-\mathrm{II}} \begin{aligned} x+4y+z&=0\\ y-z&=0\\ 0&=0\end{aligned} \longrightarrow \begin{aligned} x+4y+z&=0\\ y&=z\end{aligned}$$

The set of solutions is the line in $\mathbb{R}^3$ formed by the intersection of the planes $x+4y+z=0$ and $y=z$

These can be described as

$$\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} -4t-t\\ t\\ t\end{bmatrix} \text{ with } t\in\mathbb{R} \text{ i.e. } \begin{bmatrix} -5t\\ t\\ t\end{bmatrix} \text{ i.e. } \boxed{t\begin{bmatrix} -5\\ 1\\ 1\end{bmatrix},\ t\in\mathbb{R}}$$

1.1/26)

$$\begin{aligned} x+y-z&=2\\ x+2y+z&=3\\ x+y+(k^2-5)z&=k\end{aligned} \xrightarrow[\mathrm{III}-\mathrm{I}]{\mathrm{II}-\mathrm{I}} \begin{aligned} x+y-z&=2\\ y+2z&=1\\ (k^2-4)z&=k-2\end{aligned} \longrightarrow \text{3 cases}$$

If $k^2-4\neq 0$ (i.e. $k\neq\pm 2$):

$$\xrightarrow{\mathrm{III}\div(k^2-4)} \begin{aligned} x+y-z&=2\\ y+2z&=1\\ z&=\frac{1}{k+2}\end{aligned} \xrightarrow{\mathrm{I}-\mathrm{II}+\mathrm{III}} \begin{aligned} x&=1+\frac{1}{k+2}\\ y+2z&=1\\ z&=\frac{1}{k+2}\end{aligned}$$

$$\xrightarrow{\mathrm{II}-2\mathrm{III}} \begin{cases} x=1+\frac{1}{k+2}\\ y=1-\frac{2}{k+2}\\ z=\frac{1}{k+2}\end{cases}$$

the unique solution

If $k=2$

$$\longrightarrow \begin{aligned} x+y-z&=2\\ y+2z&=1\\ 0&=0\end{aligned} \xrightarrow{\mathrm{I}-\mathrm{II}} \begin{aligned} x-3z&=1\\ y+2z&=1\end{aligned} \longrightarrow \begin{cases} x=1+3z\\ y=1-2z\\ z \text{ free}\end{cases}$$

∞-ly many solutions

$$\begin{bmatrix} 1+3t\\ 1-2t\\ t\end{bmatrix},\ t\in\mathbb{R}$$

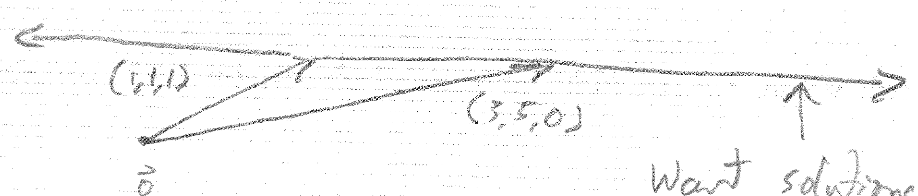
If $k=-2$

$$\longrightarrow \begin{aligned} x+y-z&=2\\ y+2z&=1\\ 0&=-4\end{aligned}$$

impossible to satisfy: no solutions

JB

1.1/42



Want solutions on this line.

$$\begin{bmatrix}3\\5\\0\end{bmatrix}-\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\4\\-1\end{bmatrix}$$ is parallel to the line so I want solutions

to look like $\begin{bmatrix}1\\1\\1\end{bmatrix}+t\begin{bmatrix}2\\4\\-1\end{bmatrix}$, $t\in\mathbb{R}$

i.e. $\begin{aligned}x&=1+3t\\ y&=1+5t\\ z&=1-t\end{aligned}$, $t\in\mathbb{R}$ Choose one variable to be free, say z,

Then $t=1-z$ so $\begin{aligned}x&=1+3(1-z)\\ y&=1+5(1-z)\\ z&\text{ free}\end{aligned}$ i.e. $\begin{cases}x=4-3z\\ y=6-5z\\ z \text{ free}\end{cases}$

The easiest set of three equations in three unknowns whose solution is $\begin{cases}x=4-3z\\ y=6-5z\\ z \text{ free}\end{cases}$

is $\boxed{\begin{cases}x+3z=4\\ y+5z=6\end{cases}}$

1.2/11

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c}1&0&2&4&-8\\0&1&-3&-1&6\\3&4&-6&8&0\\0&-1&3&4&-12\end{array}\right]\xrightarrow{\text{III}-3\text{I}}\left[\begin{array}{cccc|c}1&0&2&4&-8\\0&1&-3&-1&6\\0&4&-12&-4&24\\0&-1&3&4&-12\end{array}\right]\xrightarrow[\text{IV}+\text{II}]{\text{III}-4\text{II}}\left[\begin{array}{cccc|c}1&0&2&4&-8\\0&1&-3&-1&6\\0&0&0&0&0\\0&0&0&3&-6\end{array}\right]$$

$$\xrightarrow{\text{III}\leftrightarrow\text{IV}}\left[\begin{array}{cccc|c}1&0&2&4&-8\\0&1&-3&-1&6\\0&0&0&3&-6\\0&0&0&0&0\end{array}\right]\xrightarrow{\text{III}\div 3}\left[\begin{array}{cccc|c}1&0&2&4&-8\\0&1&-3&-1&6\\0&0&0&1&-2\\0&0&0&0&0\end{array}\right]\xrightarrow[\text{I}-4\text{III}]{\text{II}+\text{III}}\left[\begin{array}{cccc|c}1&0&2&0&0\\0&1&-3&0&4\\0&0&0&1&-2\\0&0&0&0&0\end{array}\right]$$

rref.

so x_3 free and $\begin{aligned}x_1+2x_3&=0\\ x_2-3x_3&=4\\ x_4&=-2\end{aligned}$

i.e. solutions are

$$\boxed{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2t\\4+3t\\t\\-2\end{bmatrix},\ t\in\mathbb{R}}$$

II

JB

1.2/18 a. Rows II and III <u>both</u> have leading 1's so $\boxed{\text{not rref}}$.

b. • Pivot of each row is 1 ✓

• Each column containing a leading 1 has only 0's in all other entries ✓

• Each row containing a leading 1 has leading 1's further left in each row above ✓

So it is $\boxed{\text{in rref}}$

c. Row III contains a leading 1, but Row II has <u>no</u> leading 1's. so $\boxed{\text{not rref}}$

d. • Pivot is 1 ✓

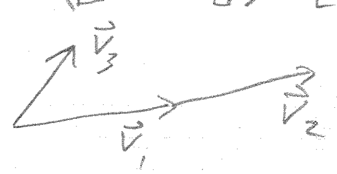
• ← these conditions don't matter because they only apply

• ↙ to matrices with more than one row ✓

so it is $\boxed{\text{in rref}}$

1.3/4 $\text{rref}\left(\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\right)=\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$ so $\boxed{\text{rank}=2}$

1.3/6



$\vec{v}_1 \parallel \vec{v}_2 \Rightarrow x\vec{v}_1+y\vec{v}_2 \parallel \vec{v}_1$ for any $x,y\in\mathbb{R}$

so since $\vec{v}_3 \nparallel \vec{v}_1$ $x\vec{v}_1+y\vec{v}_2=\vec{v}_3$ is impossible. $\boxed{\text{No solutions}}$

1.3/22 rref must look exactly like a square matrix with only 1's on the diagonal.

$$\left[\begin{array}{cccc|c}1&0&0&*&*\\0&1&0&*&*\\0&0&1&*&*\\0&0&0&0&a\end{array}\right]$$

would mean infinitely many solutions if $a=0$ or no solutions if $a\neq0$

likewise $\left[\begin{array}{cccc|c}1&*&0&0&*\\0&0&1&0&*\\0&0&0&1&*\\0&0&0&0&a\end{array}\right]$ etc.

If rref has no row of 0's, it must be <u>exactly</u>

$$\left[\begin{array}{cccc|c}1&0&0&0&*\\0&1&0&0&*\\0&0&1&0&*\\0&0&0&1&*\end{array}\right]$$

which always means unique solution.

III

JB

1.3/23 To read off a unique solution, rref must look like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1.3/58 Want $x\begin{bmatrix}1\\3\\2\end{bmatrix} + y\begin{bmatrix}2\\6\\4\end{bmatrix} + z\begin{bmatrix}-1\\-3\\-2\end{bmatrix} = \begin{bmatrix}3\\b\\c\end{bmatrix}$.

i.e. want $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & 6 & -3 & b \\ 2 & 4 & -2 & c \end{array}\right]$ to have at least one solution.

row reduce to $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & b-9 \\ 0 & 0 & 0 & c-6 \end{array}\right]$ which is only consistent when $0 = b-9$ and $0 = c-6$

i.e. $\boxed{\begin{cases} b=9 \\ c=6 \end{cases}}$. In that case any $\begin{bmatrix}x\\y\\z\end{bmatrix}$ satisfying $x+2y-z=3$ would work

IV

JB