2.1.5 By Theorem 2.1.2, the three columns of the 2×3 matrix A are $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$, so that

$$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}.$$

2.1.6 Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

2.1.7 Note that $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = [\vec{v}_1 \ldots \vec{v}_m] \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix}$, so that T is indeed linear, with matrix $[\vec{v}_1\ \vec{v}_2 \cdots$

First, suppose that $ad - bc \neq 0$ and $a \neq 0$. Let $D = ad - bc$ for simplicity. We continue our work in part (a):

$$\left[\begin{array}{l} x_1 + \frac{b}{a}x_2 = \frac{1}{a}y_1 \\ \frac{D}{a}x_2 = -\frac{c}{a}y_1 + y_2 \end{array}\right] \cdot \frac{a}{D} \rightarrow$$

$$\left[\begin{array}{l} x_1 + \frac{b}{a}x_2 = \frac{1}{a}y_1 \\ x_2 = -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array}\right] -\frac{b}{a}(II) \rightarrow$$

$$\left[\begin{array}{l} x_1 = (\frac{1}{a} + \frac{bc}{aD})y_1 - \frac{b}{D}y_2 \\ x_2 = -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array}\right]$$

$$\left[\begin{array}{l} x_1 = \frac{d}{D}y_1 - \frac{b}{D}y_2 \\ x_2 = -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array}\right]$$

(Note that $\frac{1}{a} + \frac{bc}{aD} = \frac{D+bc}{aD} = \frac{ad}{aD} = \frac{d}{D}$.)

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, as claimed. If $ad - bc \neq 0$ and $a = 0$, then we have to solve the system.

$$\left[\begin{array}{l} cx_1 + dx_2 = y_2 \\ bx_2 = y_1 \end{array}\right] \begin{array}{l} \div c \\ \div b \end{array}$$

$$\left[\begin{array}{l} x_1 + \frac{d}{c}x_2 = \frac{1}{c}y_2 \\ x_2 = \frac{1}{b}y_1 \end{array}\right] -\frac{d}{c}(II)$$

$$\left[\begin{array}{l} x_1 = -\frac{d}{bc}y_1 + \frac{1}{c}y_2 \\ x_2 = \frac{1}{b}y_1 \end{array}\right]$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (recall that $a = 0$), as claimed.

2.1.14 a By Exercise 13 $\begin{bmatrix} 2 & 3 \end{bmatrix}$

2.1.43 a $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2\ 3\ 4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The transformation is indeed linear, with matrix $[2\ 3\ 4]$.

b If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then T is linear with matrix $[v_1\ v_2\ v_3]$, as in part (a).

c Let $[a\ b\ c]$ be the matrix of T. Then $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a\ b\ c] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ does the job.

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = B\left(A \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = B \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} pa + qc \\ ra + sc \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B\left(A \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = B \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} pb + qd \\ rb + sd \end{bmatrix}$$

So, $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \left(T \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 \left(T \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \Big[= \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} pb + qd \\ rb + sd \end{bmatrix} \Big]$

$$A = \begin{bmatrix} pa+qc & pb+qd \\ ra+sc & rb+sd \end{bmatrix}$$

Write $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$: $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$. (See Figure 2.21.)

2.2.6 By Theorem 2.2.1, $\text{proj}_L\begin{bmatrix}1\\1\\1\end{bmatrix} = \left(\vec{u}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right)\vec{u}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix}1\\2\end{bmatrix}$

$\vec{u} = \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$, so that $\text{proj}_L\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{5}{3}\cdot\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}\frac{10}{9}\\\frac{5}{9}\\\frac{10}{9}\end{bmatrix}$.

2.2.17 We want, $\begin{bmatrix}a & b\\ b & -a\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}av_1 & +bv_2\\ bv_1 & -av_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$.

Now, $(a-1)v_1 + bv_2 = 0$ and $bv_1 - (a+1)v_2$, which is a system with solutions of the form $\begin{bmatrix}bt\\(1-a)t\end{bmatrix}$, where t is an arbitrary constant.

Let's choose $t = 1$, making $\vec{v} = \begin{bmatrix}b\\1-a\end{bmatrix}$.

Similarly, we want $A\vec{w} = -\vec{w}$. We perform a computation as above to reveal $\vec{w} = \begin{bmatrix}a-1\\b\end{bmatrix}$ as a possible choice. A quick check of $\vec{v}\cdot\vec{w} = 0$ reveals that they are indeed perpendicular.

Now, any vector $\vec{x}$ in $\mathbb{R}$ can be written in terms of components with respect to $L = \text{span}(\vec{v})$ as $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp} = c\vec{v} + d\vec{w}$. Then, $T(\vec{x}) = A\vec{x} = A(c\vec{v} + d\vec{w}) = A(c\vec{v}) + A(d\vec{w}) = cA\vec{v} + dA\vec{w} = c\vec{v} - d\vec{w} = \vec{x}^{||} - \vec{x}^{\perp} = \text{ref}_L(\vec{x})$, by Definition 2.2.2.

(The vectors $\vec{v}$ and $\vec{w}$ constructed above are both zero in the special case that $a = 1$ and $b = 0$. In that case, we can let $\vec{v} = \vec{e}_1$ and $\vec{w} = \vec{e}_2$ instead.)

2.2.21 $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = -\vec{e}_1$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$. (See Figure 2.26.)

2.2.25 The matrix $A = \begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$ represents a horizontal shear, and its inverse $A^{-1} = \begin{bmatrix}1 & -k\\0 & 1\end{bmatrix}$ represents such a shear as well, but "the other way."

2.2.30 Write $A = [\,\vec{v}_1 \quad \vec{v}_2\,]$; then $A\vec{x} = [\vec{v}_1 \quad \vec{v}_2]\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2$. We must choose $\vec{v}_1$ and $\vec{v}_2$ in such a way that $x_1\vec{v}_1 + x_2\vec{v}_2$ is a scalar multiple of the vector $\begin{bmatrix}1\\2\end{bmatrix}$, for all x_1 and x_2. This is the case if (and only if) both $\vec{v}_1$ and $\vec{v}_2$ are scalar multiples of $\begin{bmatrix}1\\2\end{bmatrix}$.

For example, choose $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}0\\0\end{bmatrix}$, so that $A = \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}$.