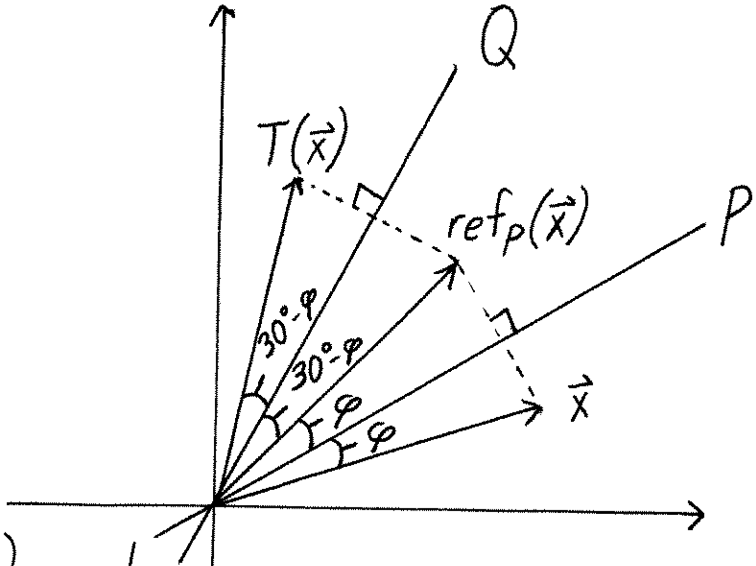
2.3 #30

a.



$\vec{x}$ and $T(\vec{x})$ and $ref_P(\vec{x})$ all have the same length.

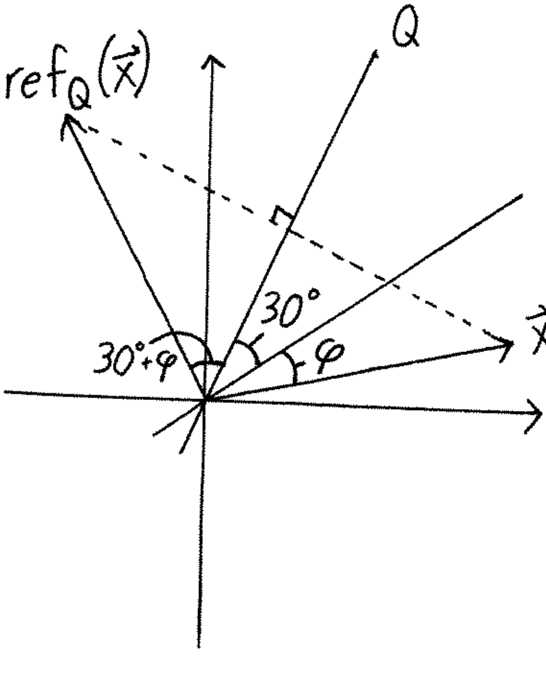
The angle enclosed by $\vec{x}$ and $T(\vec{x})$ is
$$(30°-\varphi)+(30°-\varphi)+\varphi+\varphi=60°.$$

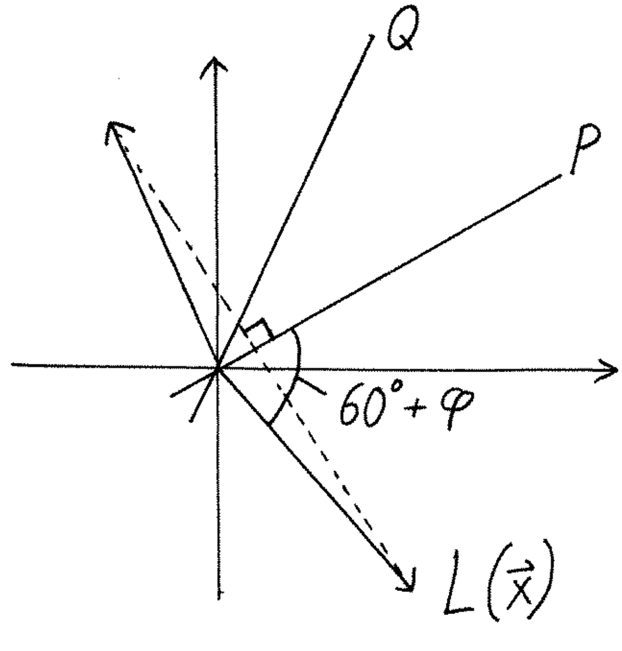
b. T is a counterclockwise rotation by $60°$.

c. By Theorem 2.2.3

$$T=\begin{pmatrix}\cos 60° & -\sin 60° \\ \sin 60° & \cos 60°\end{pmatrix}=\begin{pmatrix}1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2\end{pmatrix}$$

d.





L is a clockwise rotation by $60°$.

$$L = \begin{pmatrix} \cos -60° & -\sin -60° \\ \sin -60° & \cos -60° \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

2.3 #31

$$A = \begin{pmatrix} \text{—} \vec{r}_1 \text{—} \\ \text{—} \vec{r}_2 \text{—} \\ \vdots \\ \text{—} \vec{r}_n \text{—} \end{pmatrix} \qquad B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix}$$

$$AB = \begin{pmatrix} \text{—} \vec{r}_1 \text{—} \\ \text{—} \vec{r}_2 \text{—} \\ \vdots \\ \text{—} \vec{r}_n \text{—} \end{pmatrix} \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \cdot \vec{v}_1 & \vec{r}_1 \cdot \vec{v}_2 & \cdots & \vec{r}_1 \cdot \vec{v}_m \\ \vec{r}_2 \cdot \vec{v}_1 & \vec{r}_2 \cdot \vec{v}_2 & \cdots & \vec{r}_2 \cdot \vec{v}_m \\ \vdots & \vdots & \ddots & \vdots \\ \vec{r}_n \cdot \vec{v}_1 & \vec{r}_n \cdot \vec{v}_2 & \cdots & \vec{r}_n \cdot \vec{v}_m \end{pmatrix}$$

$$\begin{pmatrix} \text{—} \vec{r}_i \text{—} \end{pmatrix} \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \vec{r}_i \cdot \vec{v}_1 & \vec{r}_i \cdot \vec{v}_2 & \cdots & \vec{r}_i \cdot \vec{v}_m \end{pmatrix}$$

(i'th row of AB) = (i'th row of A) B

2.3 #63 Since $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ is 3×2 and I_3 is 3×3, X must be 2×3.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} a+4b & c+4d & e+4f \\ 2a+5b & 2c+5d & 2e+5f \\ 3a+6b & 3c+6d & 3e+6f \end{pmatrix}$$

Setting this equal to I_3,

$a+4b=1$	$c+4d=0$	$e+4f=0$
$2a+5b=0$	$2c+5d=1$	$2e+5f=0$
$3a+6b=0$	$3c+6d=0$	$3e+6f=1$

Solving for a and b, we find

$$\left(\begin{array}{cc|c} 1 & 4 & 1 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & -3 & -2 \\ 0 & -6 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & 1 \end{array}\right)$$

there are no solutions.

inconsistent

2.3#65

$$X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$X^2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & ab+bc \\ 0 & c^2 \end{pmatrix}$$

If $X^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then $a^2 = ab+bc = c^2 = 0.$

Thus $a=0$ and $c=0.$ $ab+bc=0$ for any real b. Thus all possible matrices are

$$X = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \qquad b \in \mathbb{R}.$$

2.3#67 Suppose $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

$$(A - I_2)^2 = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The transformation $A - I_2$ performed twice sends every vector $\vec{x} \in \mathbb{R}^2$ to the origin.

2.4#29 To be invertible the matrix needs full rank.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 3 & k^2-1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{pmatrix}$$

$$k^2-3k+2 \neq 0$$

$$(k-1)(k-2) \neq 0$$

The matrix is invertible for all values of k except 1 and 2.

2.4#34 a. To have full rank and hence be invertible, we must have $a \neq 0$ $b \neq 0$ and $c \neq 0$. If these conditions hold, then $A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$.

b. To be invertible, an $n \times n$ diagonal matrix must have no 0s on the diagonal. In other words, every diagonal element must be nonzero.

2.4#36 By Theorem 2.4.3 an $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$.

$\operatorname{rank}(A)=n$ if and only if $\operatorname{rref}(A)=I_n$.
Suppose A has been reduced to triangular form with all diagonal entries nonzero.

<u>Upper Triangular</u>

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \Longrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & 0 \\ 0 & a_{22} & \cdots & a_{2(n-1)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & a_{(n-1)(n-1)} & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \Rightarrow \cdots \Rightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

<u>Lower Triangular</u>

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \Longrightarrow \cdots \Longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

Conversely, suppose the triangular form has some diagonal entry equal to 0, say $a_{kk}=0$.

Upper Triangular

Without loss of generality, suppose $a_{jj} \neq 0$ whenever $k < j \leq n$. Restricting attention to rows k through n, we row reduce.

$$\begin{pmatrix} 0 & \cdots & 0 & a_{k(k+1)} & \cdots & a_{kn} \\ 0 & \cdots & 0 & a_{(k+1)(k+1)} & \cdots & a_{(k+1)n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & \cdots & 0 & a_{k(k+1)} & \cdots & a_{k(n-1)} & 0 \\ 0 & \cdots & 0 & a_{(k+1)(k+1)} & \cdots & a_{(k+1)(n-1)} & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & & a_{(n-1)(n-1)} & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 0 & \cdots & 0 & a_{k(k+1)} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & a_{(k+1)(k+1)} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & & 1 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \Longrightarrow \cdots \Longrightarrow \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

This matrix contains a row of zeros, so $\operatorname{rank}(A) < n$.

Lower Triangular

We now replace the above assumption. WLOG, suppose $a_{jj} \neq 0$ whenever $1 \leq j < k$. We now restrict attention to rows 1 through k.

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & & a_{(k-1)(k-1)} & 0 & \cdots & 0 \\ a_{k1} & a_{k2} & \cdots & a_{k(k-1)} & 0 & \cdots & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} {}^{c}1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ & & & a_{(k-1)(k-1)} & \cdots & 0 \\ 0 & a_{k2} & \cdots & a_{k(k-1)} & \cdots & 0 \end{pmatrix}$$

$\Downarrow$ next page

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ & & & a_{(k-1)(k-1)} & \cdots & 0 \\ 0 & 0 & \cdots & a_{k(k-1)} & \cdots & 0 \end{pmatrix} \Longrightarrow \cdots \Longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ & & & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

This matrix contains a row of zeros, so $\operatorname{rank}(A) < n$.

$\boxed{2.4\#76}$

$$A\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$\det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = 1\cdot 5 - 2\cdot 2 = 1 \neq 0$, so $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ is invertible.

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$$A = A\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 8 & -3 \\ -1 & 1 \end{pmatrix}$$

$\boxed{2.4\#78}$

$$A\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{pmatrix}$$

$$A = A\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{pmatrix}$$