**Exercise 3.1-11.** As the Example 10 of section 3.1 indicated, to find out the kernel of the linear transformation represented by the given matrix is equivalent to find out the solution to the corresponding linear system, say

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix} \vec{x} = \vec{0}$$

Let us compute its reduced row echelon form as

$$\operatorname{rref} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Therefore the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix},$$

or in other words, the kernel is the straight line spanned by vector $(-2, 3, 1, 0)^T$.

**Exercise 3.1-22.** Since the elementary operations on the matrix corresponds to manipulating the vectors *within* the image of the linear transformation associated to the matrix, we are happy to go on to compute the reduced row echelon form:

$$\operatorname{rref} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the image is

$$\begin{bmatrix} x_1 + 2x_3 \\ x_2 - x_3 \\ 0 \end{bmatrix}, \quad \forall x_1, x_2, x_3 \in \mathbb{R}^1.$$

Since we have three free variables we are certainly able to use this combination to produce any two real numbers, the image is actually

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \quad \forall a, b \in \mathbb{R}^1.$$

**Exercise 3.1-42.** As the hints in the text book explained, it is straightforward to convince yourself that the matrix $B$ which makes the second reduced-row-echelon-form consistent is, ***at least*** a matrix $B$ satisfying

$$\left\{\vec{y} \in \mathbb{R}^4 : B \cdot \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}^T = \vec{0}\right\} = \left\{\vec{y} \in \mathbb{R}^4 : \begin{bmatrix} y_1 & & -3y_3 & +2y_4 \\ & y_2 & -2y_3 & +y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} \tag{1}$$

Here you may have noticed I was using boldface to indicate the restriction on $B$ may be more than equation (1). This is because we have to make sure the first two lines of the second reduced-row-echelon-form

$$\begin{bmatrix} x_1 & & -x_3 & +8x_4 = 4y_3 & -4y_4 \\ & x_2 & +2x_3 & -2x_4 = -y_3 & +y_4 \end{bmatrix} \tag{2}$$

is is making perfect sense, or in other words solvable. But actually we don't have to worry about this too much, because by reduced-row-echelon form we have already acknowledged the face that the only circumstance which can make it unsolvable/inconsistent is the appearance of lines "0=1". Why? Take system (2) as an example. The coefficient matrix for this system is a $2 \times 4$ matrix of *full rank*. This is because if it is not full rank then we can go on using elimination to reduce it, which contradicts with the fact that reduced-row-echelon-form is the *last* step of Gauss-Jordan elimination. Furthermore, a full rank matrix means the underlying linear transformation transfers $\mathbb{R}^4$ *onto* $\mathbb{R}^2$. Recall the fact that to determine if a system is solvable amounts to say to determine the existence of pre-image of a given point in the image space. Since we know in our case the linear transformation is already an onto map, surely the corresponding system is solvable. This is why we can say the only way we can make a reduced system "inconsistent" is by adding a line "0=1".

And it is obvious that one possible[1] $B$ in (1) is

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

In case you may have lost in this jungle of arguments, we'd better re-organize our thought by write down the following relations

$$\begin{aligned} & \vec{y} \text{ is in the image of the matrix } A \\ \Longleftrightarrow & \exists \vec{x}, \text{ such that } A\vec{x} = \vec{y} \\ \Longleftrightarrow & \text{the second matrix in hints is consistent} \\ \Longleftrightarrow & (1) \end{aligned}$$

[1] how many $B$ can make (1) valid?

**Exercise 3.2-34.** By the statement in this question we know the following equation holds

$$\begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which results the following

$$\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}$$

or in other words

$$\vec{v}_4 = -\frac{1}{4}\vec{v}_1 - \frac{1}{2}\vec{v}_2 - \frac{3}{4}\vec{v}_3.$$

**Exercise 3.2-38.**

**a.** By Definition 3.2.3 on page 116, we need only to verify that the vectors $\{\vec{v}_i\}, i = 1, \ldots, m$ span the subspace $V$. For $\forall \vec{x} \in V$ since $m$ is the largest number of linearly independent vectors in $V$ we know the equation

$$\sum_{i=1}^{k} r_i \vec{v}_i + r_0 \vec{x} = \vec{0}$$

has a non-vanishing solution $(r_1, \ldots, r_m, r_0) \neq (0, \ldots, 0)$. Since $\{\vec{v}_i\}, i = 1, \ldots, m$ are linearly independent we know $r_0 \neq 0$, which means $\vec{x}$ can be written as a linear combination of $\{\vec{v}_i\}$. This concludes part (a).

**b.** By choosing a basis $\{\vec{v}_i\}, i = 1, \ldots, m$ of $V$ we know the matrix corresponds the following linear transformation

$$T(\vec{e}_i) = \vec{v}_i, 0 \leq i \leq m, \quad T(\vec{e}_i) = \vec{0}, m + 1 \leq i \leq n$$

is what we are looking for.

**Exercise 3.2-49.**We have two examples here:

$$A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}, abc \neq 0$$

and

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

**Exercise 3.3-39.** This is due to the fact that matrix multiplication does not "magnify" the rank. More concretely we have the following inequality

$$\operatorname{rank}(TX) \leq \operatorname{rank}(T).$$

An easy explanation to this fact is the rank of $T$ equals the dimension of its image[2] therefore by multiplying a matrix from the right side is doing nothing but restricting the input we feed $T$, and of course we cannot have more output. Therefore if $A = BC$ we know the rank of $A$ cannot be bigger than the rank of $B$, which at most equals 4. So $A$ cannot be invertible.

**Exercise 3.3-84.**We write $A$ and $B$ as

$$A = \begin{bmatrix} | & | & | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 \\ | & | & | & | & | & | \end{bmatrix}, B = \begin{bmatrix} | & | & | & | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 & \vec{w}_6 \\ | & | & | & | & | & | \end{bmatrix}.$$

and it is easy to see

$$\vec{v}_5 = 4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_4, \vec{w}_5 = 4\vec{w}_1 + 5\vec{w}_2 + 7\vec{w}_4$$

which reminds us the fact that the vector

$$\begin{bmatrix} 4 \\ 5 \\ 0 \\ 6 \\ -1 \\ 0 \end{bmatrix}$$

is in the kernel of $A$, but not of $B$.

**Exercise 3.4-27.** As the Definition 3.4.3 in the textbook mentioned, we know the desired matrix is

$$\begin{bmatrix} | & | & | \\ A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ 9 & 0 & 0 \\ -18 & 0 & 0 \end{bmatrix}.$$

**Exercise 3.4-53.** By the definition of coordinates we know

$$\vec{x} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 11\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 40 \\ 58 \end{bmatrix}$$

[2]one can prove this instantly by recalling the definition of rank