$\boxed{4.1}$ 20. If A is in this space, then

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

So $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ span the Space.

They are linearly independent, so they form a basis
So dimension is 3.

26. Let $p = ax^3 + bx^2 + cx + d$

$$\begin{aligned} \int_{-1}^{1} p(x)\,dx &= \int_{-1}^{1} ax^3 + bx^2 + cx + d \\ &= \tfrac{1}{4}ax^4 + \tfrac{1}{3}bx^3 + \tfrac{1}{2}cx^2 + dx \Big|_{-1}^{1} \\ &= (\tfrac{1}{4}a + \tfrac{1}{3}b + \tfrac{1}{2}c + d) - (\tfrac{1}{4}a - \tfrac{1}{3}b + \tfrac{1}{2}c - d) \\ &= \tfrac{2}{3}b + 2d \end{aligned}$$

So p is in the space when

$$\tfrac{2}{3}b + 2d = 0 \quad \text{and} \quad p(1) = 0 \quad (a+b+c+d=0)$$
$$b = -3d \qquad a+b+c+d = 0$$
$$b = -3d \qquad a + c - 2d = 0$$

2 equations and 4 variables,
so solution space is 2-dimensional
We need 2 linearly independent solutions

$a=2 \quad b=-3 \quad c=0 \quad d=1$
and $a=0 \quad b=-3 \quad c=2 \quad d=1$ work

So $p_1 = 2x^3 - 3x^2 + 1$
and $p_2 = -3x^2 + 2x + 1$ form a basis
Dimension is 2.

44. If $S = [w_1 \ w_2 \ w_3]$

then $S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = [w_1 \ w_2 \ 0]$

and $AS = [Aw_1 \ Aw_2 \ Aw_3]$

We need $Aw_1 = w_1$, $Aw_2 = w_2$, and $Aw_3 = 0$

That is, we need w_1, w_2 to be in the plane V
and we need w_3 to be normal to V.

Let v_1, v_2, v_3 be a basis of $\mathbb{R}^3$
with v_1, v_2 in V and v_3 normal to V.

We need
$$w_1 = av_1 + bv_2$$
$$w_2 = cv_1 + dv_2$$
$$w_3 = ev_3$$

that is
$$S = [av_1 + bv_2 \quad cv_1 + dv_2 \quad ev_3]$$
$$= a[v_1 \ 0 \ 0] + b[v_2 \ 0 \ 0]$$
$$+ c[0 \ v_1 \ 0] + d[0 \ v_2 \ 0] + e[0 \ 0 \ v_3]$$

$[v_1 \ 0 \ 0]$, $[v_2 \ 0 \ 0]$, $[0 \ v_1 \ 0]$,
$[0 \ v_2 \ 0]$, $[0 \ 0 \ v_3]$ form a basis

So dimension is 5.

4.2

31. $T(f+g) = \begin{pmatrix} f(0)+g(0) & f(1)+g(1) \\ f(2)+g(2) & f(3)+g(3) \end{pmatrix} = T(f) + T(g)$

$T(cf) = \begin{pmatrix} c f(0) & c f(1) \\ c f(2) & c f(3) \end{pmatrix} = c\,T(f)$

So T is linear.

$\dim P_2 = 3$ ($1, x, x^2$ is a basis)

$\dim \mathbb{R}^{2\times 2} = 4$ ($\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is a basis)

So T is not an isomorphism.

46. $T(f+g) = (t-1)\big(f(t)+g(t)\big) = T(f) + T(g)$

$T(cf) = (t-1)\; c f(t) = c\,T(f)$

So T is linear

~~The~~ If p is in the image of T

then $\cancel{p(t)}\ T(q) = (t-1)\,q(t)$ for some polynomial q

So $p(1) = 0$

Polynomials with $p(1) \neq 0$ are not in the image of T.

So image$(T) \neq P$, so T is not an isomorphism.

78. a. We have to check the 8 properties in definition 4.1) (p. 154) ☹

We use $\oplus$ instead of $+$ and $\odot$ instead of ~~·~~ scalar multiplication.

1. $(x \oplus y) \oplus z = (xy)z = x(yz) = x \oplus (y \oplus z)$ ✓
2. $x \oplus y = xy = yx = y \oplus x$
3. Neutral element is 1: $x \oplus 1 = x \cdot 1 = x$
4. The negative of x is x^{-1}: $x \oplus x^{-1} = x x^{-1} = 1$
5. $k \odot (x \oplus y) = (x \oplus y)^k = (xy)^k = x^k y^k$
   $= (k \odot x)(k \odot y) = (k \odot x) \oplus (k \odot y)$
6. $(c+k) \odot x = x^{c+k} = x^c x^k = x^c \oplus x^k = (c \odot x) \oplus (k \odot x)$
7. $c \odot (k \odot x) = (k \odot x)^c = (x^k)^c = x^{ck} = ck \odot x$
8. $1 \odot x = x^1 = x$

b. $T(x \oplus y) = T(xy) = \ln(xy)$
$= \ln x + \ln y = T(x) + T(y)$

$T(c \odot x) = T(x^c) = \ln(x^c) = c \ln x + c\,T(x)$

So it's linear

ker T: $T(x) = 0$
$\ln x = 0$
$x = 1$ (the neutral element)

image $T = \mathbb{R}$ since $T(e^x) = \ln(e^x) = x$

So T is an isomorphism

$\boxed{4.3}$ 4. $f(t) = t+1$

$tf(t) = t^2 + t$

$g(t) = (t+2)(t+k) = t^2 + (2+k)t + 2k$

Use the basis $B = \{t^2, t, 1\}$

The B- coordinates are

$$[f]_B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad [tf]_B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad [g]_B = \begin{pmatrix} 1 \\ 2+k \\ 2k \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2+k \\ 1 & 0 & 2k \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2+k \\ 0 & 1 & 1 \\ 1 & 0 & 2k \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2+k \\ 0 & 1 & 1 \\ 0 & -1 & k-2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2+k \\ 0 & 1 & 1 \\ 0 & 0 & k-1 \end{pmatrix}$$

So f, tf, g are linearly independent when $k \neq 1$

15. $B = \{1, i\}$

$T(1) = 1$ $\qquad T(i) = -i$

$[T(1)]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad [T(i)] = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

rref $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so A is invertible

So T is an isomorphism.

16. $B = \{1+i, 1-i\}$

$T(1+i) = 1-i$ $\qquad T(1-i) = 1+i$

$[T(1+i)]_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad [T(1-i)]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

rref $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so A is invertible

So T is an isomorphism

45. a. $B = \{1+i, 1-i\}$

$U = \{1, i\}$

$[1+i]_U = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad [1-i]_U = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

b. $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$SB = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$AS = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$SB = AS$

c. We need to find $[1]_B$ and $[i]_B$

$1 = \frac{1}{2}(1+i) + \frac{1}{2}(1-i)$

So $[1]_B = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$i = \frac{1}{2}(1+i) - \frac{1}{2}(1-i)$

So $[i]_B = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

Matrix is $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$