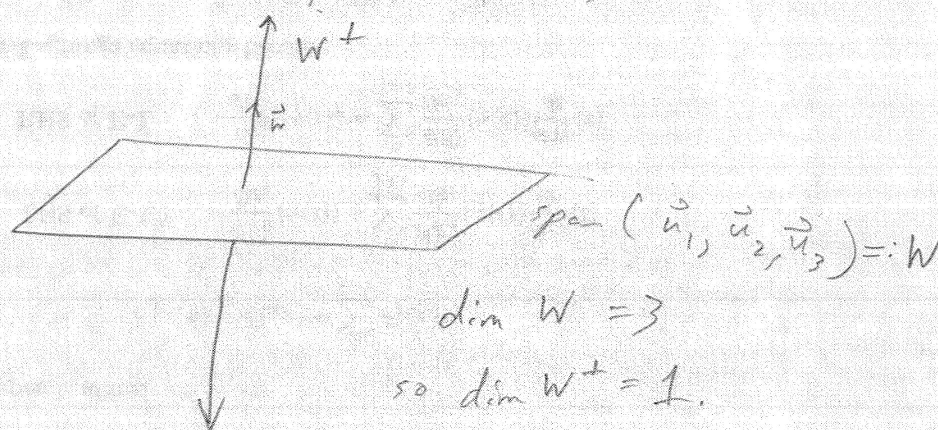
Homework 8 Solutions

5.1/16 $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are themselves orthonormal since $\|\vec{u}_1\| = \|\vec{u}_2\| = \|\vec{u}_3\| = 1$ and $\vec{u}_i \perp \vec{u}_j$ $i, j \in \{1,2,3\}$. So the question asks for a unit vector perpendicular to $\text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ in $\mathbb{R}^4$.



$\dim W = 3$

so $\dim W^\perp = 1$.

$\boxed{\text{Only two unit vectors in } W^\perp, \vec{w} \text{ and } -\vec{w}.}$

$\vec{w} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ works.

5.1/17 $W^\perp = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0 \right\}$

$= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; \begin{matrix} a + 2b + 3c + 4d = 0 \\ 5a + 6b + 7c + 8d = 0 \end{matrix} \right\}$ = solution set of $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \end{array}\right]$

$\xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow$ $a = c + d$, $b = -2c - 3d$, c free, d free $\rightsquigarrow \begin{bmatrix} c+d \\ -2c-3d \\ c \\ d \end{bmatrix} = c\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

so $\boxed{W^\perp \text{ has basis } \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)}$

I

JB

5.1/28 $\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

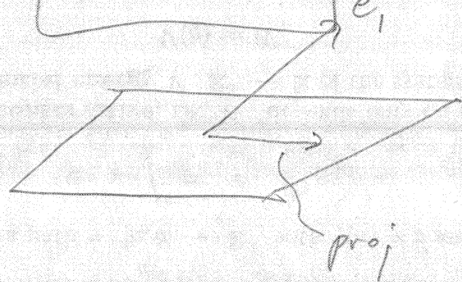
Find $\text{proj}_{\text{span}(\vec{a},\vec{b},\vec{c})}(\hat{e}_1)$. Don't know Gram-Schmidt yet, but fortunately $\vec{a},\vec{b},\vec{c}$ mutually orthogonal, so

$\underbrace{\frac{\vec{a}}{\|\vec{a}\|}}_{\vec{u}}, \underbrace{\frac{\vec{b}}{\|\vec{b}\|}}_{\vec{v}}, \underbrace{\frac{\vec{c}}{\|\vec{c}\|}}_{\vec{w}}$ is an $\underset{\text{orthonormal basis}}{\text{ONB}}$ for $\text{span}(\vec{a},\vec{b},\vec{c})$

$$\text{So } \text{proj}_{\text{span}(\vec{a},\vec{b},\vec{c})}(\hat{e}_1) = (\hat{e}_1 \cdot \vec{u})\vec{u} + (\hat{e}_1 \cdot \vec{v})\vec{v} + (\hat{e}_1 \cdot \vec{w})\vec{w}$$

$$\boxed{= \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}}$$

Double check



want $(\hat{e}_1 - \text{proj}) \perp \vec{a}, \vec{b}, \vec{c}$

$(\hat{e}_1 - \text{proj}) \cdot \vec{a} = 0$

" $\cdot \vec{b} = 0$

" $\cdot \vec{c} = 0$

II

SB

5.2/32 $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x+y+z=0 \right\}$ $\dim V = 2$ so a basis is any two non-parallel vectors. e.g. $\left(\underset{\vec{v}}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}, \underset{\vec{w}}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \right)$

To find ONB, perform Gram-Schmidt on $(\vec{v}, \vec{w})$

$$\vec{u}_1 := \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{u}_2 := \frac{\vec{w} - (\vec{u}_1 \cdot \vec{w})\vec{u}_1}{\| \quad \|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\boxed{\text{ONB: } (\vec{u}_1, \vec{u}_2)}$$

5.2/34 $\ker A$ = solution set for $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$

i.e. $\begin{cases} a = c+d \\ b = -2c-3d \\ c, d \text{ free} \end{cases}$ i.e. $\left\{ \begin{bmatrix} c+d \\ -2c-3d \\ c \\ d \end{bmatrix} = c \underset{\vec{v}}{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + d \underset{\vec{w}}{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}} \middle| \, c, d \in \mathbb{R} \right\}$

So a basis for $\ker A$ is $(\vec{v}, \vec{w})$.

ONB comes from Gram-Schmidt on

$$\vec{u}_1 := \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 := \frac{\vec{w} - (\vec{w} \cdot \vec{u}_1)\vec{u}_1}{\| \quad \|} = \frac{1}{\sqrt{102}} \begin{bmatrix} -1 \\ -4 \\ -7 \\ \frac{102}{17} \end{bmatrix}$$

$$\boxed{\text{ONB: } (\vec{u}_1, \vec{u}_2)}$$

III

5.2/36 $M = \begin{bmatrix} 1 & -1/2 & 9 \\ 1 & 7/2 & -4 \\ 1 & 7/2 & 3 \\ 1 & -1/2 & 9 \end{bmatrix}$
$\quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

Gram-Schmidt on $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$:

$$\vec{u}_1 := \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 := \frac{\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1}{\| \uparrow \|} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{u}_3 := \frac{\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2}{\| \uparrow \|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

So $\boxed{Q} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 \end{bmatrix}$

$$\boxed{R} = \begin{bmatrix} \vec{u}_1 \cdot \vec{v}_1 & \vec{u}_1 \cdot \vec{v}_2 & \vec{u}_1 \cdot \vec{v}_3 \\ 0 & \vec{u}_2 \cdot \vec{v}_2 & \vec{u}_2 \cdot \vec{v}_3 \\ 0 & 0 & \vec{u}_3 \cdot \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 17/2 \\ 0 & 4 & -19/2 \\ 0 & 0 & 7/\sqrt{2} \end{bmatrix}$$

NV
JB

5.3/37 $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$, $\vec{a} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$

Is there orthogonal T such that $\begin{cases} T(\vec{x}) = \vec{a} \\ T(\vec{y}) = \vec{b} \end{cases}$?

$\boxed{\text{No.}}$ $\vec{x} \perp \vec{y}$, but $\vec{a} \cdot \vec{b} = 6 \neq 0$ so $\vec{a} \not\perp \vec{b}$.

T orthogonal $\Rightarrow T(\vec{x}) \perp T(\vec{y})$ $\lightning$.

5.3/40 $Q = \begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix}$ $(\vec{u}_1, \vec{u}_2)$ result of Gram-Schmidt on $(\vec{v}_1, \vec{v}_2)$

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$, $\vec{u}_2$

Matrix is $\boxed{QQ^T = \frac{1}{50} \begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{bmatrix}}$

5.3/48 $A^T = QR \Rightarrow A = (A^T)^T = (QR)^T = \underbrace{R^T}_{\text{lower triangular}} \underbrace{Q^T}_{\text{orthog (still)}}$

5.3/56 $\mathbb{R}^{2\times3} \xrightarrow{L} \mathbb{R}^{3\times2}$
$A \longmapsto A^T$

Linear: $(kA+B)^T = kA^T + B^T$ ✓

$\ker L = \{A \mid A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\} = \{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\}$

Since $\dim \mathbb{R}^{2\times3} = \dim \mathbb{R}^{3\times2} = 6$, $\ker L = 0 \Rightarrow \boxed{L \text{ iso.}}$

✓ JB