Solutions - 5.4, 5.5

2) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ Im(A): $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x+y \\ x+2y \\ x+3y \end{bmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

We know that $(\text{Im } A)^\perp = \ker(A^T)$ [Theorem 5.4.1]

$\dim(\text{Im } A) + \dim((\text{Im} A)^\perp) = \dim(\mathbb{R}^3) = 3$, and $\dim(\text{Im}(A)) = 2$.

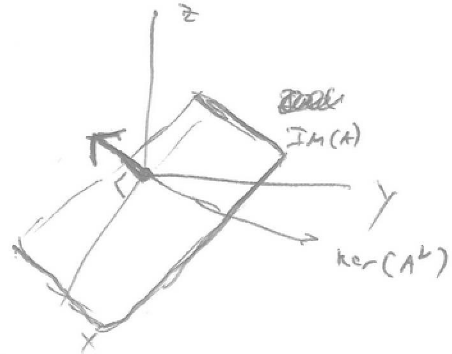
So $\dim((\text{Im} A)^\perp) = 1$. Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be in $(\text{Im } A)^\perp$.

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \vec{x} = 0 \implies x + y + z = 0$

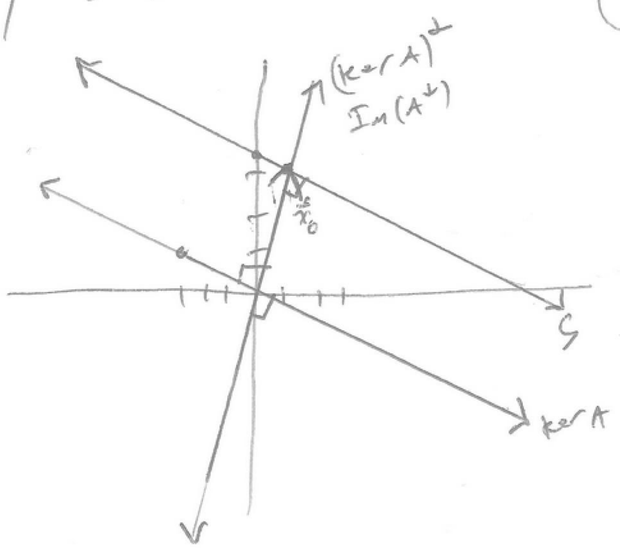
$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \vec{x} = 0 \implies x + 2y + 3z = 0$

$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ will do.

So $(\text{Im } A)^\perp = \ker(A^\perp) = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$ and $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a basis for $\ker(A^\perp)$.



9) $A\vec{x} = 0 \implies \vec{x} = t\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ for any $t \in \mathbb{R}$.



- $\ker A$ = the line $y = -\frac{x}{3}$

  $(\ker A)^\perp$ = the line $y = 3x$ (from high school?)

- $A^\perp = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\text{Im}(A^\perp) = t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

  $= (\ker A)^\perp$

- $A\vec{x} = b$

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\vec{x} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 2 & 6 & 20 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & 0 & 0 \end{array}\right]$

$x + 3y = 10$

$y = \frac{10}{3} - \frac{x}{3}$

parallel to ker A

b) $\ker(A) \perp \mathrm{Im}(A^T)$

c) $\ker(A) \parallel S$

d) See graph.

e) $\vec{x}_0$ is the shortest, see 5.4.4.
$\vec{x}_0$ is the least-squares approximation.

18) Yes. From 16, $\mathrm{rank}(A) = \mathrm{rank}(A^T)$
Apply 17 to A^T: $\mathrm{rank}(A^T) = \mathrm{rank}(A^{T^T} A^T) = \mathrm{rank}(AA^T)$
Also from 17: $\mathrm{rank}(A^TA) = \mathrm{rank}(A)$.

So $\mathrm{rank}(A) = \mathrm{rank}(A^T)$

$\Longrightarrow \mathrm{rank}(AA^T) = \mathrm{rank}(A^TA)$

19) $A\vec{x} = \vec{b}$

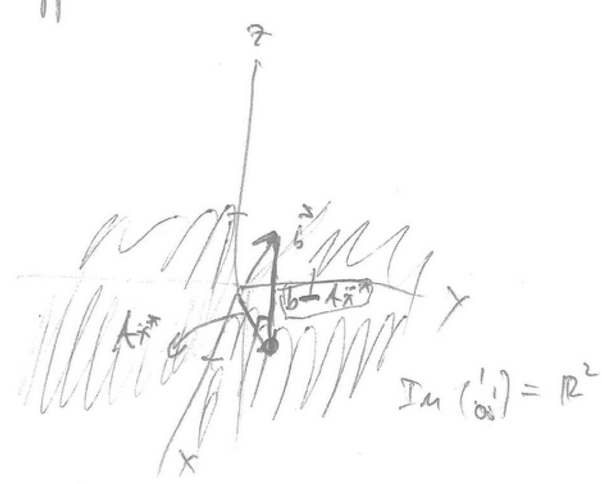
$\begin{bmatrix} 1 & \\ & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Theorem 5.4.5: $A^TA\vec{x} = A^Tb$, the normal equation, has exact solution equal to the least-squares approximation of $A\vec{x} = \vec{b}$

$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ $\qquad A^T\vec{b} = \begin{bmatrix} 1 & & 0 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$I_2 \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the least squares approximation.



24) $A\vec{x}=\vec{b}$

$S \in \mathbb{R}^{3\times 3}$, orthogonal.

Use normal equation of the following: let $\tilde{A} = SA$. Let $\tilde{b} = S\vec{b}$

$$\tilde{A}\vec{x} = \tilde{b} \xrightarrow[\text{Squares}]{\text{Least}} \tilde{A}^T\tilde{A}\vec{x} = \tilde{A}^T\tilde{b}$$

$$\longrightarrow (SA)^T SA\vec{x} = (SA)^T S\vec{b}$$

$$\longrightarrow A^T S^T S A\vec{x} = A^T S^T S\vec{b}.$$

$$\left(\begin{array}{c}\text{Recall that } S \\ \text{is orthogonal} \\ \Longleftrightarrow \\ S^T = S^{-1}\end{array}\right) \longrightarrow A^T A\vec{x} = A^T\vec{b}$$

This is the normal equation of $A\vec{x}=\vec{b}$! Therefore the least-squares solution of $SA\vec{x}=S\vec{b}$ is $\vec{x}^* = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

5.5

3) $\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T S\vec{y}$

4) Need to verify definition 5.5.1.

i) Is $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$?

$$\langle \vec{x}, \vec{y} \rangle = (S\vec{x})^T (S\vec{y}) = (S\vec{x}) \cdot (S\vec{y}) = (S\vec{y}) \cdot (S\vec{x}) = (Sy)^T (Sx) = \langle y, x \rangle \quad \checkmark$$

ii) Is $\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$?

$$\begin{aligned}\langle \vec{x}+\vec{y}, \vec{z} \rangle &= (S(\vec{x}+\vec{y}))^T (S\vec{z}) \\ &= (S\vec{x}+S\vec{y})^T (S\vec{z}) \\ &= \left[(S\vec{x})^T + (S\vec{y})^T\right](S\vec{z}) \\ &= (S\vec{x})^T(S\vec{z}) + (S\vec{y})^T(S\vec{z}) = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle \quad \checkmark\end{aligned}$$

iii) Is $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$?

$$\langle c\vec{x}, \vec{y} \rangle = \left[S(cx)\right]^T (Sy) = c\,[Sx]^T[Sy] = c\langle x, y \rangle \quad \checkmark$$

$\langle \vec{x}, \vec{x} \rangle = (S\vec{x})^T (S\vec{x}) = S\vec{x} \cdot S\vec{x} = \|S\vec{x}\| \neq 0$ as long as $\vec{x} \notin \ker(S)$.

If $\langle \, , \, \rangle$ is an inner product, then no non-zero vector is in $\ker(S)$. $\therefore$ S is invertible.

b) $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} \iff S\vec{x} \cdot S\vec{y} = \vec{x} \cdot \vec{y} \iff S$ is orthogonal.

10) Let $g \in P_2$, $g = a + bt + ct^2$

Need $\langle g, t \rangle = 0 = \frac{1}{2}\int_{-1}^{1} at + bt^2 + ct^3 \, dt = \frac{at^2}{2} + \frac{bt^3}{3} + \frac{ct^4}{4} \Big|_{-1}^{1}$

$= \frac{a}{2} + \frac{b}{3} + \frac{c}{4} - \left[\frac{a}{2} - \frac{b}{3} + \frac{c}{4}\right]$

$= \frac{2b}{3}$

$\implies b = 0$

e.g. $\langle 1, t \rangle = 0$ $\quad \langle t^2, t \rangle = 0$. But we need an orthonormal basis for $(t)^\perp$

Gram-Schmidt: $\|1\| = \sqrt{\langle 1, 1 \rangle} = \sqrt{\frac{1}{2}\int_{-1}^{1} dt} = 1$

$u_1 = 1$.

$u_2 = \dfrac{v_2 - \langle v_2, u_1 \rangle u_1}{\| v_2 - \langle v_2, u_1 \rangle u_1 \|} = \dfrac{t^2 - \left(\frac{1}{2}\int_{-1}^{1} t^2 dt\right) \cdot 1}{\| t^2 - \left(\frac{1}{2}\int_{-1}^{1} t^2 dt\right) \cdot 1 \|}$

$\frac{1}{2}\int_{-1}^{1} t^2 dt = \frac{1}{6} t^3 \Big|_{-1}^{1} = \frac{1}{6} - \frac{-1}{6} = \frac{1}{3}$.

$\| t^2 - \frac{1}{3} \| = \sqrt{\frac{1}{2}\int_{-1}^{1} (t^2 - \frac{1}{3})^2 dt}$

$= \sqrt{\frac{1}{2}\int_{-1}^{1} t^4 - \frac{2t^2}{3} + \frac{1}{9} \, dt}$

$= \sqrt{\frac{1}{2}\left(\frac{2}{5} - \frac{4}{9} + \frac{2}{9}\right)} = \sqrt{\frac{4}{45}}$

$= \frac{2}{3\sqrt{5}}$

$\therefore u_2 = \dfrac{t^2 - \frac{1}{3}}{\frac{2}{3}\sqrt{\frac{1}{5}}} = \dfrac{3\sqrt{5}t^2 - \sqrt{5}}{2}$

Basis $= \left\{ 1, \dfrac{3\sqrt{5}t^2 - \sqrt{5}}{2} \right\}$

$\measuredangle(v,w) = \cos^{-1}\frac{\langle v,w\rangle}{\|v\|\|w\|}$

$$\langle \cos(t), \cos(t+\delta)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(t+\delta)\,dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\left[\cos(t)\cos(\delta) - \sin(t)\sin(\delta)\right] dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(t)\cos(\delta)\,dt - \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(t)\sin(\delta)\,dt$$

$\cos(t)\sin(t)\sin(\delta)$ is odd, so $\int_{-\pi}^{\pi} \cos(t)\sin(t)\sin(\delta)\,dt = 0$

$$\langle \cos(t), \cos(t+\delta)\rangle = \frac{1}{\pi}\cos(\delta)\int_{-\pi}^{\pi}\cos^2(t)\,dt = \frac{\cos(\delta)}{\pi}\int_{-\pi}^{\pi}\frac{1+\cos 2x}{2}\,dx$$

$$= \frac{\cos(\delta)}{\pi}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)\Big|_{-\pi}^{\pi} = \cos(\delta) \qquad (\text{Neat!})$$

16) a) Let $v_1 = 1$. $\|v_1\|^2 = \int_0^1 1^2\,dt = 1$, so $u_1 = v_1$

$v_2 = t$. $v_2 - \langle v_2, u_1\rangle u_1 = t - \int_0^1 t\,dt = t - \frac{1}{2}$

$$\|v_2 - \langle v_2, u_1\rangle u_1\| = \|t - \tfrac{1}{2}\|^2 = \int_0^1 \left(t-\tfrac{1}{2}\right)^2 = \int_0^1 t^2 - t + \tfrac{1}{4}\,dt = \frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4}\Big|_0^1$$
$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

So $u_2 = \dfrac{t - \frac{1}{2}}{\sqrt{1/12}} = 2\sqrt{3}\,t - \sqrt{3}$

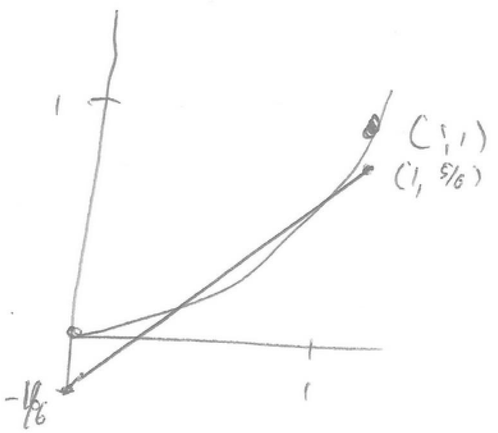
Basis: $\{1, 2\sqrt{3}\,t - \sqrt{3}\}$

b) We want $\text{proj}_{P_1}(t^2) = \langle 1, t^2\rangle 1 + \langle 2\sqrt{3}t - \sqrt{3}, t^2\rangle (2\sqrt{3}t - \sqrt{3})$

$$\langle 1, t^2\rangle = \int_0^1 t^2\,dt = \frac{1}{3}$$

$$\langle 2\sqrt{3}t - \sqrt{3}, t^2\rangle = \sqrt{3}\langle 2t-1, t^2\rangle = \sqrt{3}\langle 2t, t^2\rangle - \sqrt{3}\langle 1, t^2\rangle$$
$$= \sqrt{3}\int_0^1 2t^3\,dt - \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{6}$$

$$\therefore \text{proj}_{P_1}(t^2) = \frac{1}{3} + \frac{\sqrt{3}}{6}(2\sqrt{3}t - \sqrt{3}) = \frac{1}{3} + t - \frac{1}{2} = t - 1/6$$



20) $\langle v, w\rangle = v^T \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} w$

a) Want $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{v}$ such that $\langle \vec{v}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\rangle = 0$

$$\langle v, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\rangle = [x\ y]\begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [x\ y]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = x + 2y = 0$$
$$x = -2y \implies \text{e.g. } v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

All vectors: $t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\forall t \in \mathbb{R}$.

b) We know that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\rangle = [1\ 0]\begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$= [1\ 0]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1$ ✓

$$\langle \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\rangle = [-2\ 1]\begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = [0\ 4]\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 4$$

So $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\frac{1}{4}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an orthonormal basis.