6.1#14 $\det\begin{pmatrix} 4 & 0 & 0 \\ 3 & k & 0 \\ 2 & 1 & 0 \end{pmatrix} = 0$ for all k. Thus the matrix is not invertible for any k.

6.1#30 $\det\begin{pmatrix} 4-\lambda & 2 & 0 \\ 4 & 6-\lambda & 0 \\ 5 & 2 & 3-\lambda \end{pmatrix} = (3-\lambda)\left[(4-\lambda)(6-\lambda)-8\right]$

$= (3-\lambda)(\lambda^2-10\lambda+24-8) = (3-\lambda)(2-\lambda)(8-\lambda) = 0$

$\underline{\lambda = 2, 3, 8}$

6.1#39 $\det\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix} = (-1)^n(\text{Prod } P)$ where P is the only non-zero pattern and n is the number of inversions in that pattern.

$$\text{Prod } P = 5\cdot 4\cdot 2\cdot 3\cdot 1 = 120$$

$$n = 4+2+1+1+0 = 8$$

$$\det A = (-1)^8(120) = 120$$

6.2#18 We fix a basis of P_2: $\{1, t, t^2\}$.

$$T(f(t)) = f(3t-2)$$

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = T(1) = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ First Column.

$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = T(t) = 3t-2 = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ Second Column.

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = T(t^2) = (3t-2)^2 = 9t^2 - 12t + 4 = \begin{pmatrix}4\\-12\\9\end{pmatrix}$$ Third Column.

$$\det(T) = \det\begin{pmatrix}1 & -2 & 4\\0 & 3 & -12\\0 & 0 & 9\end{pmatrix} = 1\cdot 3\cdot 9 = 27$$

6.2 #28 $V = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} : x_1 + 2x_2 + 3x_3 = 0\right\}$. We fix a basis of V: $\left\{\begin{pmatrix}-3\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$.

$$T(\vec{v}) = \begin{pmatrix}1\\2\\3\end{pmatrix} \times \vec{v}.$$

$$T\begin{pmatrix}1\\0\end{pmatrix}_V = \begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}-3\\0\\1\end{pmatrix} = \det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\1 & 2 & 3\\-3 & 0 & 1\end{pmatrix} = \begin{pmatrix}2\\-10\\6\end{pmatrix}$$

$$\begin{pmatrix}2\\-10\\6\end{pmatrix} = 6\begin{pmatrix}-3\\0\\1\end{pmatrix} - 10\begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}6\\-10\end{pmatrix}_V$$ First Column.

$$T\begin{pmatrix}0\\1\end{pmatrix}_V = \begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}-2\\1\\0\end{pmatrix} = \det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\1 & 2 & 3\\-2 & 1 & 0\end{pmatrix} = \begin{pmatrix}-3\\-6\\5\end{pmatrix}$$

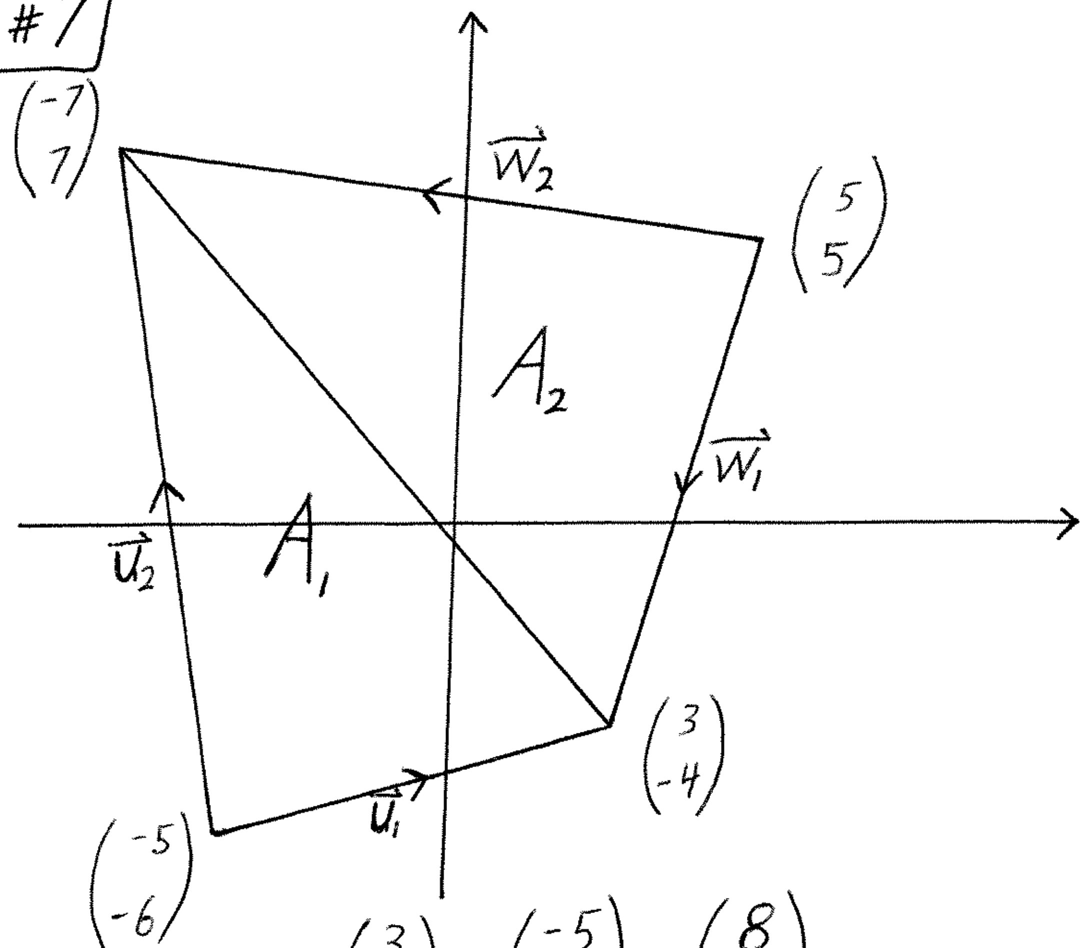
$$\begin{pmatrix}-3\\-6\\5\end{pmatrix} = 5\begin{pmatrix}-3\\0\\1\end{pmatrix} - 6\begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}5\\-6\end{pmatrix}_V$$ Second Column.

$$\det(T) = \det\begin{pmatrix}6 & 5\\-10 & -6\end{pmatrix} = -36 + 50 = 14$$

6.2 #38 $\det(A^T) = \det(A)$.

$$\det(A^TA) = \det(A^T)\det(A) = (\det(A))^2 = 3^2 = 9.$$

6.3 #7



$$\vec{u}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} -5 \\ -6 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix} -7 \\ 7 \end{pmatrix} - \begin{pmatrix} -5 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 13 \end{pmatrix}$$

$$A_1 = \frac{1}{2}\left|\det\left(\vec{u}_1 \middle| \vec{u}_2\right)\right| = \frac{1}{2}\left|\det\begin{pmatrix} 8 & -2 \\ 2 & 13 \end{pmatrix}\right| = 54$$

$$\vec{w}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -9 \end{pmatrix}$$

$$\vec{w}_2 = \begin{pmatrix} -7 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -12 \\ 2 \end{pmatrix}$$

$$A_2 = \frac{1}{2}\left|\det\left(\vec{w}_1 \middle| \vec{w}_2\right)\right| = \frac{1}{2}\left|\det\begin{pmatrix} -2 & -12 \\ -9 & 2 \end{pmatrix}\right| = 56$$

$$\text{Area} = A_1 + A_2 = 54 + 56 = 110$$

6.3#11 We cannot draw any conclusions about $\det(A)$ unless we also assume $\vec{v}_1$ and $\vec{v}_2$ are nonzero. With this additional assumption, we first show $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Suppose $\vec{v}_1$ and $\vec{v}_2$ are dependent.

Then $\vec{v}_2 = c\vec{v}_1$.

$$T(\vec{v}_2) = T(c\vec{v}_1) = c\,T(\vec{v}_1) = c\,3\vec{v}_1 = 3\vec{v}_2 \neq 4\vec{v}_2.$$

This is a contradiction, so $\vec{v}_1$ and $\vec{v}_2$ must be linearly independent.

Thus, $V = \{\vec{v}_1, \vec{v}_2\}$ is a basis of $\mathbb{R}^2$.

Let B be the matrix of T with respect to the basis V. Then

$B\begin{pmatrix}1\\0\end{pmatrix} = T(\vec{v}_1) = 3\vec{v}_1 = \begin{pmatrix}3\\0\end{pmatrix}$ First Column.

$B\begin{pmatrix}0\\1\end{pmatrix} = T(\vec{v}_2) = 4\vec{v}_2 = \begin{pmatrix}0\\4\end{pmatrix}$ Second Column.

$B = \begin{pmatrix}3 & 0\\0 & 4\end{pmatrix}$. For some invertible change of basis matrix S, $B = S^{-1}AS$.

$$\det(B) = \det(S^{-1})\det(A)\det(S) = \det(A).$$

$$\det(A) = \det(B) = 12$$

6.3#14

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$A = \left(\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3 \right) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{pmatrix}$$

$$\det\left(A^T A\right) = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 9 \\ 0 & 9 & 29 \end{pmatrix} = 3 \cdot 29 - 9 \cdot 9 = 6$$

$$V\left(\vec{v}_1, \vec{v}_2, \vec{v}_3\right) = \sqrt{\det\left(A^T A\right)} = \sqrt{6}$$

6.3#24

$$\begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ -1 \end{pmatrix}$$

$$\det \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 0 & -9 & 7 \end{pmatrix} = 2(28+45) = 146$$

$$\det \begin{pmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{pmatrix} = \det \begin{pmatrix} \boxed{8} & 3 & 0 \\ 0 & \boxed{4} & 26 \\ -1 & 0 & \boxed{7} \end{pmatrix} = (-1)^0 (8)(4)(7)$$

$$+ (-1)^2 (-1)(3)(26) = 146$$

$$\det\begin{pmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{pmatrix} = \det\begin{pmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 0 & -25 & 7 \end{pmatrix} = 2(21+125) = 292$$

$$\det\begin{pmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{pmatrix} = \det\begin{pmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 0 & -9 & -25 \end{pmatrix} = 2(-100+27) = -146$$

$$x = \frac{146}{146} = 1$$

$$y = \frac{292}{146} = 2$$

$$z = \frac{-146}{146} = -1$$

To check this result,

$$\begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+6 \\ 8-5 \\ 6-7 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ -1 \end{pmatrix} \quad \checkmark$$