7.1-7 By the deifnition of eigenvector we know that

$$\vec{v} \in \ker(A - \lambda I_n)$$

and since the kernel of matrix $A - \lambda I_n$ is not trivial we know it is not invertible

7.1-8 Solving the following equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix},$$

we know that all matrices in the form of

$$\begin{bmatrix} 5 & b \\ 0 & d \end{bmatrix}$$

are the desied matrices.

7.1-18 It is easy to see that for the reflection R about a plane V we have

$$\begin{aligned} R\vec{v} &= 1 \cdot \vec{v}, \quad \forall \vec{v} \in V \\ R\vec{w} &= -1 \cdot \vec{w}, \quad \forall \vec{w} \perp V^{\cdot} \end{aligned}$$

We can choose any basis of V and an orthogonormal complement of V as the basis of corresponding eigenspaces.

7.1-37 For part (a), at first we can easily see that this linear transformation is the composition of a scaling 5 and a reflection

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix}$$

Therefore by the exercise 18 we know the eigenvalues are $5, -5$

Part (b) is similar to exercise 2.2-17 from the textbook. It is easy to see that this reflection is about the line $2y = x$, therefore we can easily find two eigenvectors

$$\begin{bmatrix} 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \end{bmatrix}.$$

7.1-41 Assume we have in general 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the requirement on A is nothing but

$$\begin{array}{r} a+b=c+d \\ a+2b=c+2d \end{array} \Longrightarrow b=d, a=c$$

Therefore in general

$$A = a \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

7.2-12 This question is just pure calculation:

$$\det \begin{vmatrix} 2-\lambda & -2 & 0 & 0 \\ 1 & -1-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & -4 \\ 0 & 0 & 2 & -3-\lambda \end{vmatrix} = \lambda(\lambda-1)^2(\lambda+1)$$

Therefore the eigenvalues are $0, +1, -1$.

7.2-32 Again we do the following calculation at first

$$\det \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ k & 3 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + k$$

Now let us analyze the property of function $g(\lambda) = \lambda^3 - 3\lambda$. Since $g'(\lambda) = 3(\lambda^2 - 1)$ has two zeros, -1 and 1, we know this function has two local extremums, $g(-1) = 2$ and $g(1) = -2$. Therefore by drawing the graph of function $g(\lambda)$ we know that as long as we have $-2 < k < 2$ the graph will intersect with x-axis at three distinct points.

7.2-40 Recall the way we compute the product of two matrices:

$$\mathrm{tr}(AB) = \sum_{i=0}^{n} (\sum_{j=0}^{n} a_{ij} b_{ji}) = \sum_{j=0}^{n} (\sum_{i=}^{n} b_{ij} a_{ji}) = \mathrm{tr}(BA)$$

7.2-42 By exercise 7.2-40 we have

$$\mathrm{tr}((A+B)^2) = \mathrm{tr}(A^2 + B^2) + 2\mathrm{tr}(BA) = \mathrm{tr}(A^2) + \mathrm{tr}(B^2)$$

7.2-47 Assume

$$M = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix}$$

then we have

$$AM = \begin{bmatrix} A\vec{v_1} & A\vec{v_2} \end{bmatrix} = \begin{bmatrix} 2\vec{v_1} & 3\vec{v_2} \end{bmatrix}$$

Therefore, the existence of such non-zero matrices M is equivalent to the fact that A has 2 and/or 3 as eigenvalues.