SOLUTION KEY TO THE LINEAR ALGEBRA
FINAL EXAM

(1) We find a least squares solution to

\[ A\vec{x} = \vec{y} \quad \text{or} \quad \begin{bmatrix} 1 & -2 & (-2)^2 \\ 1 & -1 & (-1)^2 \\ 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]

The normal equation is

\[ A^T A\vec{x} = A^T \vec{y} = \vec{y} \quad \text{or} \quad \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} a_* \\ b_* \\ c_* \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ -17 \end{bmatrix}. \]

The least-squares solution is

\[ \vec{x} = \begin{bmatrix} a_* \\ b_* \\ c_* \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0 \\ 9 \\ -5 \end{bmatrix}, \]

so the sought-after polynomial is

\[ p(t) = \frac{9}{10} t - \frac{1}{2} t^2. \]

(2) (a)

\[ \text{rref}(A) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \]

So a basis for \( V = \text{Im}(A) \) is given by the first two columns of \( A \). A routine application of the Gram-Schmidt process to these two columns yields the orthonormal basis

\[ \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}. \]

Another basis is

\[ \left\{ \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}. \]
(b) Since $A$ is a projection matrix onto $V$, $\text{Ker}(A) = V^\perp$.

From (1), the vector $\begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$ is a basis for $\text{Ker}(A)$ so an orthonormal basis consists of the vector $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

(c) 

$$P = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}.$$ 

On the other hand, it is geometrically obvious that $\vec{x} = \text{proj}_V \vec{x} + \text{proj}_{V^\perp} \vec{x}$ for any vector $\vec{x} \in \mathbb{R}^n$ and subspace $V \subset \mathbb{R}^n$, which in our case can be read to say $A + P = I_3$, providing a second (and easier) way of computing $P$.

(3) (a) The ellipse is $q(\vec{x}) = 1$ where $q(\vec{x}) = \vec{x}^T A \vec{x}$ and 

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}.$$ 

We have $p_A(\lambda) = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2)$ so the eigenvalues of $A$ are $\lambda_1 = 7, \lambda_2 = 2$. The principal axes are

- $c_1$ axis: $E_7 = \text{Ker}(7I - A) = \text{span} \vec{u}_1, \quad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $c_2$ axis: $E_2 = \text{Ker}(2I - A) = \text{span} \vec{u}_2, \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(b) In $c_1$-$c_2$ coordinates: $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$ so the equation of the ellipse becomes

$$7c_1^2 + 2c_2^2 = 1.$$ 

(c) The lengths of the semiaxes of the ellipse are $1/\sqrt{\lambda_1} = 1/\sqrt{7}$ and $1/\sqrt{\lambda_2} = 1/\sqrt{2}$.

(4) (a) We need to prove that

- $q(\vec{x}) = \langle \vec{x}, \vec{x} \rangle > 0$ for any $\vec{x} \neq \vec{0}$.

The determinants of the principal submatrices of $A$ are $\det A^{(1)} = \det[2] = 2 > 0$ and $\det A^{(2)} = \det A = 6 > 0$ so $q$ is a positive definite quadratic form and the property above holds.

(b) Let us agree that $||\vec{v}||$ denotes not the Euclidean (usual) length of $\vec{v}$ but rather the length computed using the inner
Figure 1. The ellipse $6x_1^2 + 4x_1x_2 + 3x_2^2 = 1$ with its principal axes and the vectors $\vec{u}_1/\sqrt{7}$ (black) and $\vec{u}_2/\sqrt{2}$ (blue).

product: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{q(\vec{v})}$. $E$ is not an orthonormal basis of $\mathbb{R}^2$ since, for instance, $\|\vec{e}_1\| = \sqrt{q(\vec{e}_1)} = \sqrt{2} \neq 1$. We apply the Gram-Schmidt process to the standard basis $E$ and let

$$
\vec{v}_1 = \frac{1}{\|\vec{e}_1\|} \vec{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

$$
\vec{v}_2 = \vec{e}_2 - \langle \vec{v}_1, \vec{e}_2 \rangle \vec{v}_1 = \vec{e}_2 + \sqrt{2} \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

$$
\|\vec{v}_2\| = \sqrt{q(\vec{v}_2)} = \sqrt{3}
$$

$$
\vec{v}_2 = \frac{1}{\sqrt{3}} \vec{v}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},
$$

so an orthonormal basis is $U = \{\vec{v}_1, \vec{v}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Another orthonomal basis consists of the semiaxis vectors $\lambda_1^{-1/2} \vec{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\lambda_2^{-1/2} \vec{u}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for the ellipse $q(\vec{x}) = 1$.

(5) (a) True. The equality of the eigenvalues follows from the equality of the characteristic polynomials. Since $p_M(\lambda) = \ldots$
\(\lambda^2 - (\text{Trace } M)\lambda + \det M\) for any \(2 \times 2\) matrix \(M\), it suffices to show that \(AB\) and \(BA\) have the same trace (we know this) and determinant. However, \(\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)\).

(b) True. \(A\) will be a reflection in a line \(L \subset \mathbb{R}^2\) if \(A\vec{x} = \vec{x}\) for any \(\vec{x} \in L\) and \(A\vec{x} = -\vec{x}\) for any \(\vec{x} \perp L\). Now, for our \(A\) we have \(p_A(\lambda) = \lambda^2 - 1\), so the eigenvalues of \(A\) are \(\pm 1\) (each with multiplicity one). All we need to show now is that the eigenspaces \(E_{\pm 1}\) are perpendicular lines in \(\mathbb{R}^2\) for then \(A\) will be the reflection in \(L = E_{\pm 1}\). There is no need to find the eigenvectors for \(A\) explicitly since, \(A\) being real and symmetric, its eigenspaces are necessarily orthogonal.

(c) False. If there were such a basis \(\mathcal{B}\) then the matrices \(A = [T]_{\mathcal{E}}\) and \(I_5 = [T]_{\mathcal{B}}\) would satisfy

\[I_5 = S^{-1}AS\]

where \(S\) is the matrix of change of basis \(\mathcal{E} \to \mathcal{B}\). However, it follows from the equation above that \(A = SI_5S^{-1} = SS^{-1} = I_5\), so \(A\) was the identity matrix to begin with. Hence the statement is false unless \(T\) is the identity transformation of \(\mathbb{R}^5\).

(d) True. Take a basis (actually, any spanning set of vectors will do just as well) for \(V\), say \(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}\). Then \(\text{Im}(A) = V\) where

\[
A = \begin{bmatrix}
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k
\end{bmatrix}.
\]

(e) Since \(A, B\) are positive definite, for any nonzero \(\vec{x} \in \mathbb{R}^n\), \(\vec{x}^T A\vec{x} > 0\) and \(\vec{x}^T B\vec{x} > 0\). Adding the two equations we obtain \(\vec{x}^T (A+B)\vec{x} = \vec{x}^T A\vec{x} + \vec{x}^T B\vec{x} > 0\) so \(A+B\) is positive definite as well.

(f) False. If \(A^T\vec{b} = \vec{0}\) and \(A\vec{x} = \vec{b}\) is consistent then \(\vec{b}\) is both in \(\text{Ker}(A^T)\) and in \(\text{Im}(A)\). Since these are mutually orthogonally complementary subspaces, this would imply \(\vec{b} = \vec{0}\). In other words, if \(A^T\vec{b} = \vec{0}\) then system \(A\vec{x} = \vec{b}\) is necessarily inconsistent unless \(\vec{b} = \vec{0}\)!

(g) True. Under a suitable change of basis, the matrix \(B\) of the shear will be of the form

\[
B = \begin{bmatrix}
1 & t \\
0 & 1
\end{bmatrix}
\]
for some number $t$. This is because, if \( \{ \vec{v}_1, \vec{v}_2 \} \) is a basis of \( \mathbb{R}^2 \) with \( \vec{v}_1 \) in the line \( L \) of the shear, then \( A\vec{v}_1 = \vec{v}_1 \) (hence the first column of \( B \)) and \( A\vec{v}_2 - \vec{v}_2 \) must be some vector \( t\vec{v}_1 \in L \). A quick calculation shows that

\[
B^2 + I_2 = 2B.
\]

If \( S \) is the matrix of change of basis, then \( SBS^{-1} = A \). Multiplying equation (2) by \( S \) on the left and \( S^{-1} \) on the right we obtain the desired result.