SOLUTION KEY TO THE LINEAR ALGEBRA FINAL EXAM

(1) We find a least squares solution to

$$A\vec{x} = \vec{y} \qquad \text{or} \qquad \begin{bmatrix} 1 & -2 & (-2)^2 \\ 1 & -1 & (-1)^2 \\ 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The normal equation is

$$A^T A\vec{x}_* = A^T\vec{y} = \vec{y}_* \qquad \text{or} \qquad \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} a_* \\ b_* \\ c_* \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ -17 \end{bmatrix}.$$

The least-squares solution is

$$\vec{x}_* = \begin{bmatrix} a_* \\ b_* \\ c_* \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0 \\ 9 \\ -5 \end{bmatrix}$$

so the sought-after polynomial is $p(t) = \frac{9}{10}t - \frac{1}{2}t^2$.

(2) (a)

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

So a basis for $V = \text{Im}(A)$ is given by the first two columns of A. A routine application of the Gram-Schmidt process to these two columns yields the orthonormal basis $\left\{ \frac{1}{3\sqrt{2}} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Another basis is $\left\{ \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$.

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(b) Since A is a projection matrix onto V, $\operatorname{Ker}(A) = V^{\perp}$. From (1), the vector $\begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$ is a basis for $\operatorname{Ker}(A)$ so an orthonormal basis consists of the vector $\frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$.

(c)

$$P = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}.$$

On the other hand, it is geometrically obvious that $\vec{x} = \operatorname{proj}_V \vec{x} + \operatorname{proj}_{V^{\perp}} \vec{x}$ for any vector $\vec{x} \in \mathbb{R}^n$ and subspace $V \subset \mathbb{R}^n$, which in our case can be read to say $A + P = I_3$, providing a second (and easier) way of computing P.

(3) (a) The ellipse is $q(\vec{x}) = 1$ where $q(\vec{x}) = \vec{x}^T A \vec{x}$ and

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}.$$

We have $p_A(\lambda) = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2)$ so the eigenvalues of A are $\lambda_1 = 7, \lambda_2 = 2$. The principal axes are

$$c_1 \text{ axis: } E_7 = \operatorname{Ker}(7I - A) = \operatorname{span} \vec{u}_1, \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$c_2 \text{ axis: } E_2 = \operatorname{Ker}(2I - A) = \operatorname{span} \vec{u}_2, \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

(b) In c_1-c_2 coordinates: $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$ so the equation of the ellipse becomes

$$7c_1^2 + 2c_2^2 = 1.$$

(c) The lengths of the semiaxes of the ellipse are $1/\sqrt{\lambda_1} = 1/\sqrt{7}$ and $1/\sqrt{\lambda_2} = 1/\sqrt{2}$.

(4) (a) We need to prove that

- $q(\vec{x}) = \langle \vec{x}, \vec{x} \rangle > 0$ for any $\vec{x} \neq \vec{0}$.

The determinants of the principal submatrices of A are $\det A^{(1)} = \det[2] = 2 > 0$ and $\det A^{(2)} = \det A = 6 > 0$ so q is a positive definite quadratic form and the property above holds.

(b) Let us agree that $\|\vec{v}\|$ denotes not the Euclidean (usual) length of $\vec{v}$ but rather the length computed using the inner

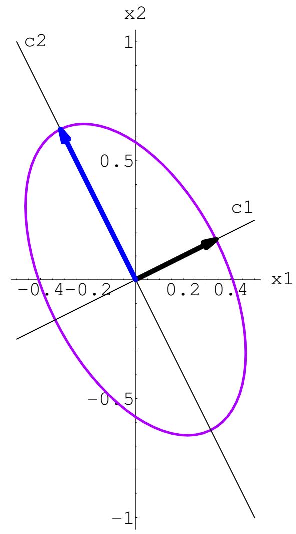

FIGURE 1. The ellipse $6x_1^2 + 4x_1x_2 + 3x_2^2 = 1$ with its principal axes and the vectors $\vec{u}_1/\sqrt{7}$ (black) and $\vec{u}_2/\sqrt{2}$ (blue).

product: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{q(\vec{v})}$. $\mathfrak{E}$ is not an orthonormal basis of $\mathbb{R}^2$ since, for instance, $\|\vec{e}_1\| = \sqrt{q(\vec{e}_1)} = \sqrt{2} \neq 1$. We apply the Gram-Schmidt process to the standard basis $\mathfrak{E}$ and let

$$\vec{v}_1 = \frac{1}{\|\vec{e}_1\|}\vec{e}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\tilde{v}_2 = \vec{e}_2 - \langle \vec{v}_1, \vec{e}_2 \rangle \vec{v}_1 = \vec{e}_2 + \sqrt{2}\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\|\tilde{v}_2\| = \sqrt{q(\tilde{v}_2)} = \sqrt{3}$$

$$\vec{v}_2 = \frac{1}{\sqrt{3}}\tilde{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\end{bmatrix},$$

so an orthonormal basis is $\mathfrak{U} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\end{bmatrix} \right\}$. Another orthonormal basis consists of the semiaxis vectors $\lambda_1^{-1/2}\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$ and $\lambda_2^{-1/2}\vec{u}_2 = \frac{1}{\sqrt{30}}\begin{bmatrix}1\\-2\end{bmatrix}$ for the ellipse $q(\vec{x}) = 1$.

(5) (a) *True.* The equality of the eigenvalues follows from the equality of the characteristic polynomials. Since $p_M(\lambda) =$

$\lambda^2 - (\operatorname{Trace} M)\lambda + \det M$ for any 2×2 matrix M, it suffices to show that AB and BA have the same trace (we know this) and determinant. However, $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

(b) *True.* A will be a reflection in a line $L \subset \mathbb{R}^2$ if $A\vec{x} = \vec{x}$ for any $\vec{x} \in L$ and $A\vec{x} = -\vec{x}$ for any $\vec{x} \perp L$. Now, for our A we have $p_A(\lambda) = \lambda^2 - 1$, so the eigenvalues of A are ± 1 (each with multiplicity one). All we need to show now is that the eigenspaces $E_{\pm 1}$ are perpendicular lines in $\mathbb{R}^2$ for then A will be the reflection in $L = E_{+1}$. There is no need to find the eigenvectors for A explicitly since, A being real and symmetric, its eigenspaces are necessarily orthogonal.

(c) *False.* If there were such a basis $\mathfrak{B}$ then the matrices $A = [T]_{\mathfrak{E}}$ and $I_5 = [T]_{\mathfrak{B}}$ would satisfy

$$I_5 = S^{-1}AS$$

where S is the matrix of change of basis $\mathfrak{E} \to \mathfrak{B}$. However, it follows from the equation above that $A = SI_5S^{-1} = SS^{-1} = I_5$, so A was the identity matrix to begin with. Hence the statement is false unless T is the identity transformation of $\mathbb{R}^5$.

(d) *True.* Take a basis (actually, any spanning set of vectors will do just as well) for V, say $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$. Then $\operatorname{Im}(A) = V$ where

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix}.$$

(e) Since A, B are positive definite, for any nonzero $\vec{x} \in \mathbb{R}^n$, $\vec{x}^T A\vec{x} > 0$ and $\vec{x}^T B\vec{x} > 0$. Adding the two equations we obtain $\vec{x}^T(A+B)\vec{x} = \vec{x}^T A\vec{x} + \vec{x}^T B\vec{x} > 0$ so $A+B$ is positive definite as well.

(f) *False.* If $A^T\vec{b} = \vec{0}$ and $A\vec{x} = \vec{b}$ is consistent then $\vec{b}$ is both in $\operatorname{Ker}(A^T)$ and in $\operatorname{Im}(A)$. Since these are mutually orthogonally complementary subspaces, this would imply $\vec{b} = 0$. In other words, if $A^T\vec{b} = \vec{0}$ then system $A\vec{x} = \vec{b}$ is necessarily inconsistent unless $\vec{b} = \vec{0}$!

(g) *True.* Under a suitable change of basis, the matrix B of the shear will be of the form

$$B = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

for some number t. This is because, if $\{\vec{v}_1, \vec{v}_2\}$ is a basis of $\mathbb{R}^2$ with $\vec{v}_1$ in the line L of the shear, then $A\vec{v}_1 = \vec{v}_1$ (hence the first column of B) and $A\vec{v}_2 - \vec{v}_2$ must be some vector $t\vec{v}_1 \in L$. A quick calculation shows that

$$B^2 + I_2 = 2B. \tag{2}$$

If S is the matrix of change of basis, then $SBS^{-1} = A$. Multiplying equation (2) by S on the left and S^{-1} on the right we obtain the desired result.