SOLUTION KEY TO THE LINEAR ALGEBRA MAKE-UP FINAL EXAM

(1) We find a least squares solution to

$$A\vec{x} = \vec{y} \qquad \text{or} \qquad \begin{bmatrix} 1 & \sin 0 & \cos 0 \\ 1 & \sin \pi/2 & \cos \pi/2 \\ 1 & \sin \pi & \cos \pi \\ 1 & \sin 3\pi/2 & \cos 3\pi/2 \\ 1 & \sin 2\pi & \cos 2\pi \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The normal equation is

$$A^T A\vec{x}_* = A^T\vec{y} = \vec{y}_* \qquad \text{or} \qquad \begin{bmatrix} 5 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_* \\ b_* \\ c_* \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}.$$

The least-squares solution is

$$\vec{x}_* = \begin{bmatrix} a_* \\ b_* \\ c_* \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

so the sought-after trigonometric polynomial is $p(t) = 1+2\cos t$.

(2) (a) The hyperbola is $q(\vec{x}) = 1$ where $q(\vec{x}) = \vec{x}^T A\vec{x}$ and

$$A = \frac{1}{8}\begin{bmatrix} 7 & -3 \\ -3 & -1 \end{bmatrix}.$$

We have $p_A(\lambda) = \lambda^2 - (3/4)\lambda - 1/4 = (\lambda-1)(\lambda+1/4)$ so the eigenvalues of A are $\lambda_1 = +1, \lambda_2 = -1/4$. The principal axes are

$$c_1 \text{ axis: } E_{+1} = \operatorname{Ker}(1I - A) = \operatorname{span} \vec{u}_1, \quad \vec{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$c_2 \text{ axis: } E_{-1/4} = \operatorname{Ker}\left(-\frac{1}{4}I - A\right) = \operatorname{span} \vec{u}_2, \quad \vec{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

(b) In c_1-c_2 coordinates: $q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$ so the equation of the hyperbola becomes

$$c_1^2 - \frac{1}{4}c_2^2 = 1.$$

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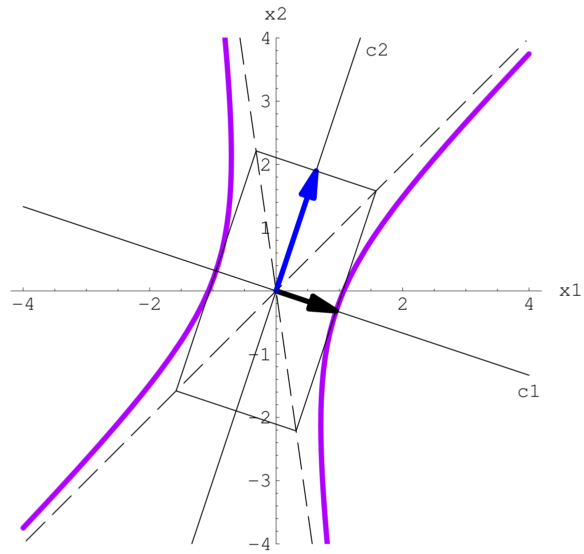

FIGURE 1. The hyperbola $7x_1^2 - 6x_1x_2 - x_2^2 = 8$ with its principal axes and the vectors $\vec{u}_1$ (black) and $2\vec{u}_2$ (blue).

(c) The asymptote are spanned by the vectors $\vec{d}_\pm = |\lambda_1|^{-1/2}u_1 \pm |\lambda_2|^{-1/2}u_2$, namely

$$\vec{d}_+ = \vec{u}_1 + 2\vec{u}_2 = \sqrt{\frac{5}{2}}\begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad \vec{d}_- = \vec{u}_1 - 2\vec{u}_2 = \sqrt{\frac{1}{10}}\begin{bmatrix}1\\-7\end{bmatrix}.$$

(3) (a) $p_A(\lambda) = \det(\lambda I - A) = \lambda^3 - 2\lambda^2 - 2\lambda = \lambda(\lambda^2 - 2\lambda - 2)$. The roots of the quadratic factor are the complex numbers $1 \pm i$ so the eigenvalues are $\lambda_1 = 0, \lambda_2 = 1 + i, \lambda_3 = 1 - i$.

(b)

$$E_0 = \text{Ker}(0I - A) = \text{span}\begin{bmatrix}1\\1\\1\end{bmatrix} \tag{1}$$

$$E_{1\pm i} = \text{Ker}((1 \pm i)I - A) = \text{span}\begin{bmatrix}0\\1\\\pm i\end{bmatrix}, \tag{2}$$

so

$$S = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & i & -i\end{bmatrix}, \quad D = \begin{bmatrix}0 & 0 & 0\\0 & 1+i & 0\\0 & 0 & 1-i\end{bmatrix}$$

and $S^{-1}AS = D$ is the desired diagonalization of A.

(4) (a) Direct calculation shows $L(E_{11}) = E_{11}$, $L(E_{22}) = E_{22}$, $L(E_{12}) = L(E_{12}) = (E_{12} + E_{21})/2$, so

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Since

$$\operatorname{rref}(M) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has pivots in the first, second and fourth columns it follows that $\operatorname{Im}(M)$ has as basis the corresponding columns of M, so we may take $\mathfrak{B} = \{E_{11}, (E_{12} + E_{21})/2, E_{22}\}$. In fact, changing this basis just a bit we obtain an orthonormal basis $\{E_{11}, (E_{12} + E_{21})/\sqrt{2}, E_{22}\}$.

(c) Since L is an orthogonal projection, $\mathcal{S}^\perp = \operatorname{Ker}(L)$, so we start by finding a basis for $\operatorname{Ker}(M)$. From $\operatorname{rref}(M)$ it is easy to read off the basis consisting of the single vector $[0, -1, 1, 0]$, hence $\operatorname{Ker} L$ has a basis consisting of the matrix $N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Since $\|N\|^2 = \operatorname{Trace}(N^T N) = 2$, the desired orthonormal basis is

$$\mathfrak{U} = \{U\}, \quad U = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(d)

$$P(A) = \langle N, A \rangle N = \frac{1}{2} \operatorname{Trace}(N^T A) N = \frac{1}{2} \begin{bmatrix} 0 & a_{12} - a_{21} \\ a_{21} - a_{12} & 0 \end{bmatrix}$$
$$= \frac{1}{2}(A - A^T).$$

Another, more direct, way of deducing this is as follows. It is geometrically obvious that $A = \operatorname{proj}_{\mathcal{S}} A + \operatorname{proj}_{\mathcal{S}^\perp} A$ for any matrix $A \in \mathbb{R}^{2\times 2}$ and subspace $\mathcal{S} \subset \mathbb{R}^{2\times 2}$ (think of adding the projections of a vector onto two mutually orthogonal lines in $\mathbb{R}^2$), which in our case can be read to say $L(A) + P(A) = A$, so $P(A) = A - (1/2)(A + A^T) = (1/2)(A - A^T)$.

(5) (a) *False.* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a counterexample.

(b) *False.* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a counterexample.

(c) *False.* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are symmetric, yet $AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not.

(d) *True.* If A is diagonalizable, it is similar to a diagonal matrix, hence *a fortiori* to a symmetric matrix.

(e) *True.* Since $A^2 = 0$ then $A(A\vec{x}) = \vec{0}$, so any vector $A\vec{x} \in \mathrm{Im}(A)$ is in $\mathrm{Ker}(A)$, therefore $\mathrm{Im}(A) \subset \mathrm{Ker}(A)$. Hence, $\dim(\mathrm{Im}(A)) \leq \dim(\mathrm{Ker}(A))$ so the rank r and the nullity n of A satisfy $r \leq n$. Since $r + n = 10$ by the rank-nullity theorem, $r \leq 5$.

(f) *True.* By the fundamental theorem of linear algebra, $\mathrm{Ker}(A)$ and $\mathrm{Im}(A^T)$ are mutually orthogonally complementary subspaces. Since $A = A^T$, $\mathrm{Im}(A^T) = \mathrm{Im}(A)$ and any two vectors $\vec{x} \in \mathrm{Ker}(A)$ and $\vec{y} \in \mathrm{Im}(A)$ must be orthogonal.

(g) *False.* $S^T AS = D$ for some orthogonal S and diagonal D. Squaring the equation and recalling that S and S^T are the inverses of one another we obtain

$$D^2 = (S^T AS)^2 = S^T ASS^T AS = S^T AIAS = S^T A^2 S = S^T 0S = 0$$

since $A^2 = 0$. By looking at the diagonal of D we conclude that $\lambda^2 = 0$ for any eigenvalue λ of A, hence D itself is the zero matrix. Therefore $A = SDS^T = S0S^T = 0$.