1. (15 points) Find the trigonometric polynomial \( p(t) = a + b \sin t + c \cos t \) of degree 1 which best fits the data:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>0</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
</tr>
<tr>
<td>( 3\pi/2 )</td>
<td>0</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>0</td>
</tr>
</tbody>
</table>
2. *(15 points)* For the hyperbola

\[ 7x_1^2 - 6x_1x_2 - x_2^2 = 8 \]

find:

(a) the principal axes,

(b) the equation of the hyperbola in the coordinate system given by the principal axes, and

(c) the asymptotes. *[Hint. The asymptotes of a hyperbola \( q(\vec{x}) = 1 \) are the diagonals of the rectangle whose vertices are]*

\[
\begin{bmatrix}
    c_1 \\
    c_2
\end{bmatrix} = \begin{bmatrix}
    \frac{\pm 1}{\sqrt{|\lambda_1|}} \\
    \frac{\pm 1}{\sqrt{|\lambda_2|}}
\end{bmatrix}
\]

in \( c_1-c_2 \) coordinates (principal axes coordinates).]
(This page intentionally left blank.)
3. (15 points) Consider the matrix

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
-2 & 1 & 1 \\
0 & -1 & 1
\end{bmatrix}
\]

(a) Find all the (real or complex) eigenvalues of \(A\).

(b) Diagonalize the matrix \(A\) (over the complex numbers, if necessary).
4. (20 points) Consider the space $\mathbb{R}^{2 \times 2}$ of $2 \times 2$ matrices. Recall that the standard basis of $\mathbb{R}^{2 \times 2}$ is given by $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$.

Consider also the linear transformation $L : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ given by

$$L(A) = \frac{1}{2}(A + A^T).$$

(a) Find the matrix $M = [L]_\mathcal{E}$ ($M$ is symmetric!).

(b) Assume as a fact that the linear transformation $L$ is the orthogonal projection\(^1\) onto some subspace $\mathcal{S} \subset \mathbb{R}^{2 \times 2}$. Find a basis $\mathcal{B}$ for $\mathcal{S}$. [Hint. $L$ must be the orthogonal projection onto its own image, so start by finding a basis for $\text{Im}(M)$.]

(c) Find an orthonormal basis $\mathcal{U}$ for $\mathcal{S}^\perp$ (the orthogonal complement of the subspace $\mathcal{S}$ with respect to the inner product in $\mathbb{R}^{2 \times 2}$). [Hint. Since $L$ is an orthogonal projection, $\mathcal{S}^\perp = \text{Ker}(L)$, so start by finding a basis for $\text{Ker}(M)$].

(d) Write down a formula for the orthogonal projection $P : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ onto $\mathcal{S}^\perp$. [Hint. The orthonormal basis $\mathcal{U}$ of (c) might help.]

---

\(^1\)With respect to the inner product $\langle A, B \rangle = \text{Trace}(A^T B)$ in $\mathbb{R}^{2 \times 2}$. 

5
5. **TRUE OR FALSE.** (5 points each) Justify your answers!

(a) If all the (real or complex) eigenvalues of $A$ are zero, then $A$ is the zero matrix.

(b) If $A$ is a (square) skew-symmetric matrix, then $\det(A) = 0$. 
(c) If $A, B$ are symmetric matrices, so is their product $AB$.

(d) If $A_{n \times n}$ is diagonalizable (over the real numbers) then $A$ is similar to a symmetric matrix.
(e) If $A^2 = 0$ for a $10 \times 10$ matrix $A$, then the inequality $\text{rank}(A) \leq 5$ must hold. [Hint. Justify the following first: for such an $A$ we have $\text{Im}(A) \subset \text{Ker}(A)$. After that, the rank-nullity theorem may help.]

(f) If $A$ is a symmetric matrix and $\vec{x} \in \text{Ker}(A)$, $\vec{y} \in \text{Im}(A)$ then $\vec{x} \perp \vec{y}$. [Hint. Use the fundamental theorem of linear algebra, or else do a direct calculation.]
(g) There is a symmetric matrix $A$ such that $A \neq 0$ and $A^2 = 0$. 
[Hint. How does $A$ being symmetric help?]