LINEAR ALGEBRA – FINAL EXAM SOLUTIONS

I. True/false.

1 . FALSE. It is true if A is symmetric, but not true in general.

2 . TRUE. $A^T\vec{v}\cdot\vec{w} = \vec{v}\cdot A\vec{w}$. So if $\vec{w}\in\ker A$, then $\vec{w}\perp$ image A^T, and vice versa.

3 . TRUE. Since A is symmetric, there is an orthogonal S that diagonalizes A. So since $S^{-1} = S^T$, $SAS^{-1} = SAS^T$ is diagonal.

4 . FALSE. The two matrices have different traces.

II. Short answer.

5 . If A is $m\times n$, then $A:\mathbb{R}^n\to\mathbb{R}^m$, so the rank/nullity theorem states that

$$\dim\ker A + \text{rank } A = n\ .$$

6 . $\text{proj}_V\vec{x} = (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + (\vec{x}\cdot\vec{v}_2)\vec{v}_2 + (\vec{x}\cdot\vec{v}_3)\vec{v}_3$.

7 . TRUE. Since A is symmetric, there is an invertible matrix S such that

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A. Now the assumption is that:

$$\left(S^{-1}AS\right)^k = S^{-1}A^kS = 0\ .$$

On the other hand,

$$\left(S^{-1}AS\right)^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n^k \end{pmatrix}.$$

So all the eigenvalues are zero, and A must be zero.

8 . The eigenvalues must be ±1. For if $A\vec{v} = \lambda\vec{v}$ for a nonzero vector $\vec{v}$, then since $A^{-1} = A^T = A$, we have

$$\vec{v} = A^{-1}A\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}\ .$$

Since $\vec{v}\neq 0$, $\lambda^2 = 1$, so $\lambda = \pm1$.

9 . To write $q(\vec{x}) = \vec{x}\cdot A\vec{x}$, set $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$.

III. Problems.

10 . (also listed as # 9 on the distributed exam) For part (i), the characteristic polynomial is:

$$p_A(\lambda) = \det \begin{pmatrix} \lambda+1 & 3 & 3 \\ -3 & \lambda-5 & -3 \\ 1 & 1 & \lambda-1 \end{pmatrix} = (\lambda-1)(\lambda-2)^2 \ .$$

(ii) Hence, the eigenvalues are 1 and 2.
(iii) The eigenspace associated to the eigenvalue 1 is:

$$E_1 = \ker \begin{pmatrix} 2 & 3 & 3 \\ -3 & -4 & -3 \\ 1 & 1 & 0 \end{pmatrix} \ .$$

From the last row, we see $x_1 = -x_2$. From the second (or first) row, we see $3x_3 = -3x_1 - 4x_2 = x_1$. So

$$E_1 = \operatorname{span} \left\{ \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} \right\} \ .$$

Similarly, for the eigenspace associated to the eigenvalue 2:

$$E_2 = \ker \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 1 & 1 & 1 \end{pmatrix} \ .$$

All the rows imply the same equation $x_1 + x_2 + x_3 = 0$. For example,

$$E_2 = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} \ .$$

11 . (listed as # 10 on the distributed exam) For part (i)

$$\operatorname{rref} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

For part (ii), the leading variables are x_1, x_2, and x_4. So the image is spanned by the corresponding columns of B. That is,

$$\text{image } B = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 8 \\ 4 \end{pmatrix} \right\} \ .$$

12 . (i) Since the dimension of $\mathbb{R}^{2\times 2}$ is 4, we only need to show that $\mathcal{B}$ is linearly independent. If we had a linear dependence, there would numbers a, b, c, d so that

$$a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4 = \begin{pmatrix} a+c & b \\ -b+c & a+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

But this says $b = 0$, which implies $c = 0$, which implies $a = 0$, which implies $d = 0$. In other words, the only such linear combination is trivial, so $\mathcal{B}$ is linearly independent.
(ii) We need to express $T\sigma_3$ in terms of the basis $\mathcal{B}$. That is

$$T\sigma_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix} = a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4 = \begin{pmatrix} a+c & b \\ -b+c & a+d \end{pmatrix} .$$

So we see $b = 0$, which implies $c = 2$, which implies $a = 1$, which implies $d = -1$. Thus, the third column of $[T]_{\mathcal{B}}$ is

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} .$$