I. True/false.

1. FALSE. It is true if $A$ is symmetric, but not true in general.

2. TRUE. $A^T \vec{v} \cdot \vec{w} = \vec{v} \cdot A \vec{w}$. So if $\vec{w} \in \ker A$, then $\vec{w} \perp \text{image } A^T$, and vice versa.

3. TRUE. Since $A$ is symmetric, there is an orthogonal $S$ that diagonalizes $A$. So since $S^{-1} = S^T$, $SAS^{-1} = SAS^T$ is diagonal.

4. FALSE. The two matrices have different traces.

II. Short answer.

5. If $A$ is $m \times n$, then $A : \mathbb{R}^n \to \mathbb{R}^m$, so the rank/nullity theorem states that

$$\dim \ker A + \text{rank } A = n.$$ 

6. $\text{proj}_V \vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + (\vec{x} \cdot \vec{v}_3)\vec{v}_3$.

7. TRUE. Since $A$ is symmetric, there is an invertible matrix $S$ such that

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. Now the assumption is that:

$$(S^{-1}AS)^k = S^{-1}A^kS = 0.$$ 

On the other hand,

$$(S^{-1}AS)^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^k \end{pmatrix}.$$ 

So all the eigenvalues are zero, and $A$ must be zero.

8. The eigenvalues must be $\pm 1$. For if $A\vec{v} = \lambda \vec{v}$ for a nonzero vector $\vec{v}$, then since $A^{-1} = A^T = A$, we have

$$\vec{v} = A^{-1}A\vec{v} = \lambda A\vec{v} = \lambda^2 \vec{v}.$$ 

Since $\vec{v} \neq 0$, $\lambda^2 = 1$, so $\lambda = \pm 1$.

9. To write $q(\vec{x}) = \vec{x} \cdot A\vec{x}$, set $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$. 

III. Problems.

10. (also listed as # 9 on the distributed exam) For part (i), the characteristic polynomial is:

\[ p_A(\lambda) = \det \begin{pmatrix} \lambda + 1 & 3 & 3 \\ -3 & \lambda - 5 & -3 \\ 1 & 1 & \lambda - 1 \end{pmatrix} = (\lambda - 1)(\lambda - 2)^2. \]

(ii) Hence, the eigenvalues are 1 and 2.
(iii) The eigenspace associated to the eigenvalue 1 is:

\[ E_1 = \ker \begin{pmatrix} 2 & 3 & 3 \\ -3 & -4 & -3 \\ 1 & 1 & 0 \end{pmatrix}. \]

From the last row, we see \( x_1 = -x_2 \). From the second (or first) row, we see \( 3x_3 = -3x_1 - 4x_2 = x_1 \). So

\[ E_1 = \text{span} \left\{ \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} \right\}. \]

Similarly, for the eigenspace associated to the eigenvalue 2:

\[ E_2 = \ker \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 1 & 1 & 1 \end{pmatrix}. \]

All the rows imply the same equation \( x_1 + x_2 + x_3 = 0 \). For example,

\[ E_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}. \]

11. (listed as # 10 on the distributed exam) For part (i)

\[ \text{rref} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

For part (ii), the leading variables are \( x_1, x_2, \) and \( x_4 \). So the image is spanned by the corresponding columns of \( B \). That is,

\[ \text{image } B = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 8 \\ 4 \end{pmatrix} \right\}. \]
12. (i) Since the dimension of $\mathbb{R}^{2\times2}$ is 4, we only need to show that $B$ is linearly independent. If we had a linear dependence, there would be numbers $a, b, c, d$ so that

$$a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4 = \begin{pmatrix} a + c \\ -b + c \end{pmatrix} \begin{pmatrix} b \\ a + d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

But this says $b = 0$, which implies $c = 0$, which implies $a = 0$, which implies $d = 0$. In other words, the only such linear combination is trivial, so $B$ is linearly independent.

(ii) We need to express $T\sigma_3$ in terms of the basis $B$. That is

$$T\sigma_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix} = a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4 = \begin{pmatrix} a + c \\ -b + c \end{pmatrix} \begin{pmatrix} b \\ a + d \end{pmatrix}.$$

So we see $b = 0$, which implies $c = 2$, which implies $a = 1$, which implies $d = -1$. Thus, the third column of $[T]_B$ is

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$