LINEAR ALGEBRA FINAL EXAM

DECEMBER 20, 2001

Please attempt all the problems and show all your work. Don't hesitate to ask me for clarification on any questions you may have. You may **not** *use any notes, books or calculators.*

I. Answer the following questions true or false (5 points each):

1 . Let A be an $n \times n$ matrix, and suppose $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

2 . Let A be an $m \times n$ matrix. Then $(\ker A)^\perp = \text{image } A^T$.

3 . Let A be a symmetric $n \times n$ matrix. Then there is an invertible matrix S such that SAS^T is diagonal.

4 . The following two matrices are similar:

$$\begin{pmatrix} 1 & 4 & 22 & 5 \\ 9 & 1 & -3 & -1 \\ 3 & 4 & 0 & 8 \\ 0 & -11 & 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 10 & 1 & 3 \\ 0 & -3 & -3 & -1 \\ 2 & 13 & 1 & 8 \\ 4 & 12 & 0 & 1 \end{pmatrix}$$

II. Short answer (7 points each):

5 . What is the relationship between the dimension of the kernel and the rank of an $m \times n$ matrix?

6 . Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be orthonormal vectors in $\mathbb{R}^n$ spanning a subspace V. Write down a formula for $\text{proj}_V \vec{x}$, the orthogonal projection of $\vec{x} \in \mathbb{R}^n$ to the subspace V, in terms of the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

7 . Decide whether the following statement is true or false: If A is a symmetric $n \times n$ matrix and there is some positive integer k so that $A^k = 0$, then A must be the zero matrix. Give a full explanation of your answer.

8 . If A is an $n \times n$ matrix that is both symmetric and orthogonal, what are the possible eigenvalues of A? Give a full explanation of your answer.

9 . What is the matrix associated to the quadratic form: $q(x, y) = 2x^2 + 6xy + 4y^2$?

III. Problems (15 points each):

10 . Let

$$A = \begin{pmatrix} -1 & -3 & -3 \\ 3 & 5 & 3 \\ -1 & -1 & 1 \end{pmatrix} .$$

(i) Calculate the characteristic polynomial of A.

(ii) Find the eigenvalues of A.

(iii) Find a basis for each eigenspace of A.

11 . Let

$$B = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix} .$$

(i) Find the reduced row echelon form of B.

(ii) Find a basis for the image of B.

12 . Let $\mathbb{R}^{2\times 2}$ denote the four dimensional space of 2×2 matrices. Define:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(i) Show that $\mathcal{B} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a basis for $\mathbb{R}^{2\times 2}$.

(ii) Consider the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined as follows: If A is a 2×2 matrix, then

$$T(A) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} A .$$

Find the third column of the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$.