I. Answer the following questions true or false (5 points each):

1. Let $A$ be an $n \times n$ matrix, and suppose $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of $A$ with distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

2. Let $A$ be an $m \times n$ matrix. Then $(\ker A)^\perp = \text{image } A^T$.

3. Let $A$ be a symmetric $n \times n$ matrix. Then there is an invertible matrix $S$ such that $SAS^T$ is diagonal.

4. The following two matrices are similar:

\[
\begin{pmatrix}
1 & 4 & 22 & 5 \\
9 & 1 & -3 & -1 \\
3 & 4 & 0 & 8 \\
0 & -11 & 3 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
2 & 10 & 1 & 3 \\
0 & -3 & -3 & -1 \\
2 & 13 & 1 & 8 \\
4 & 12 & 0 & 1
\end{pmatrix}
\]

II. Short answer (7 points each):

5. What is the relationship between the dimension of the kernel and the rank of an $m \times n$ matrix?

6. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be orthonormal vectors in $\mathbb{R}^n$ spanning a subspace $V$. Write down a formula for $\text{proj}_V \vec{x}$, the orthogonal projection of $\vec{x} \in \mathbb{R}^n$ to the subspace $V$, in terms of the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

7. Decide whether the following statement is true or false: If $A$ is a symmetric $n \times n$ matrix and there is some positive integer $k$ so that $A^k = 0$, then $A$ must be the zero matrix. Give a full explanation of your answer.

8. If $A$ is an $n \times n$ matrix that is both symmetric and orthogonal, what are the possible eigenvalues of $A$? Give a full explanation of your answer.

9. What is the matrix associated to the quadratic form: $q(x, y) = 2x^2 + 6xy + 4y^2$?
III. Problems (15 points each):

10. Let

\[ A = \begin{pmatrix} -1 & -3 & -3 \\ 3 & 5 & 3 \\ -1 & -1 & 1 \end{pmatrix} . \]

(i) Calculate the characteristic polynomial of \( A \).
(ii) Find the eigenvalues of \( A \).
(iii) Find a basis for each eigenspace of \( A \).

11. Let

\[ B = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix} . \]

(i) Find the reduced row echelon form of \( B \).
(ii) Find a basis for the image of \( B \).

12. Let \( \mathbb{R}^{2 \times 2} \) denote the four dimensional space of \( 2 \times 2 \) matrices. Define:

\[ \sigma_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \]
\[ \sigma_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \sigma_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

(i) Show that \( B = \{ \sigma_1, \sigma_2, \sigma_3, \sigma_4 \} \) is a basis for \( \mathbb{R}^{2 \times 2} \).
(ii) Consider the linear transformation \( T : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2} \) defined as follows: If \( A \) is a \( 2 \times 2 \) matrix, then

\[ T(A) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} A . \]

Find the third column of the matrix \([T]_B\) of \( T \) with respect to the basis \( B \).