Exam #2, Linear Algebra, Spring, 2003, W. Stephen Wilson

Name: ______________________________

TA Name and section: ______________________________

NO CALCULATORS.

We work in $C[a,b]$, the continuous functions on $[a,b]$, for various a and b. We have an inner product, $< f, g >= \int_a^b f(x)g(x)dx$. Recall that P_n is the subspace of polynomials of degree $\leq n$.

I recommend being very careful and being sure you are right. You should check your work if possible. Work the problems you think are easiest first.

(1) (3 points) Show the set $V \subset P_3 \subset C[-1,1]$ defined as all $f \in P_3$ with $f(-1) = 0$ and $\int_{-1}^1 f(x)dx = 0$ is a linear subspace of P_3.

If f and g are in V we must show that $f + g$ and kf are in V.
$(f+g)(x) = f(x) + g(x)$ so $(f+g)(-1) = f(-1) + g(-1) = 0 + 0 = 0$.
$(kf)(x) = kf(x)$ so $(kf)(-1) = kf(-1) = k0 = 0$.
Likewise, $\int_{-1}^1 (f+g)(x)dx = \int_{-1}^1 f(x)dx + \int_{-1}^1 g(x)dx = 0 + 0 = 0$
and $\int_{-1}^1 kf(x)dx = k\int_{-1}^1 f(x)dx = k0 = 0$.

(2) (5 points) Find a basis for the linear space in (1).

Elements look like $f(x) = ax^3 + bx^2 + cx + d$.

$$0 = f(-1) = a(-1)^3 + b(-1)^2 + c(-1) + d = -a + b - c + d.$$

$$0 = \int_{-1}^1 f(x)dx = \int_{-1}^1 (ax^3 + bx^2 + cx + d)dx = ax^4/4 + bx^3/3 + cx^2/2 + dx]_{-1}^1 = 2b/3 + 2d.$$

So

$$b = -3d \qquad \text{and} \qquad c = -a + b + d = -a - 2d.$$

So our elements are

$$ax^3 - 3dx^2 - (a + 2d)x + d = a(x^3 - x) - d(3x^2 + 2x - 1).$$

Our basis is:

$$\{x^3 - x, 3x^2 + 2x - 1\}$$

although there are many choices.

This series of problems will analyze the function x^2 by finding its linear approximation using least squares. DO NOT USE CALCULUS III to solve the problem, i.e., no partial derivatives.

We work in $P_1 \subset P_2 \subset C[0,1]$ with the usual inner product.

(3) (3 points) Find an orthogonal (doesn't have to be orthonormal) basis for P_1.

We know we have a basis $\{1, x\}$. 1 has length 1 as $< 1, 1 >= \int_0^1 dx = 1$.

Next we evaluate $\text{proj}_{P_0}(x) =< x, 1 > 1 = \int_0^1 x dx = 1/2$ so $x - 1/2$ is orthogonal to 1.

Our orthogonal basis is: $\{1, x - 1/2\}$.

(4) (3 points) Find an orthonormal basis for P_1.

All we have to do is divide $x - 1/2$ by its length.

The square of the lenth is $< x - 1/2, x - 1/2 >= \int_0^1 (x - 1/2)^2 dx = 1/12$. So we divide $x - 1/2$ by $1/2\sqrt{3}$ to get $2\sqrt{3}x - \sqrt{3}$.

Our orthonormal basis is: $\{1, 2\sqrt{3}x - \sqrt{3}\}$.

(5) (5 points) What is the orthogonal projection of $x^2 \in P_2$ onto $P_1 \subset P_2$, i.e. $\text{proj}_{P_1}(x^2)$? (Show work.)

$$\text{proj}_{P_1}(x^2) =< x^2, 1 > 1+ < x^2, 2\sqrt{3}x - \sqrt{3} > (2\sqrt{3}x - \sqrt{3}) =$$

$$\int_0^1 x^2 dx + (\int_0^1 x^2(2\sqrt{3}x - \sqrt{3}) dx)(2\sqrt{3}x - \sqrt{3}) =$$

$$x - 1/6.$$

(6) (3 points) Using the basis $\{x^2, x, 1\}$ for P_2, find the 3×3 matrix for $\text{proj}_{P_1} : P_2 \to P_1 \subset P_2$.

We have $x^2 \to x - 1/6$, $x \to x$, and $1 \to 1$.

The matrix is

$$\left(\text{proj}_{P_1}(x^2), \text{proj}_{P_1}(x), \text{proj}_{P_1}(1)\right)$$

which is, from the above,

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1/6 & 0 & 1 \end{pmatrix}.$$

(7) (3 points) What is the dimension of the kernel of this linear transformation (in problem (6))? (Explain.)

The dimension of P_2 is 3 and the image is P_1 which has dimension 2. Because $\dim P_2 = \dim \text{Image} + \dim \text{kernel}$ we get 3=2+1, so the answer is 1.

There are other ways to do this.

(8) (3 points) Find a polynomial basis for the kernel in (7).

We know that $z - \text{proj}_{P_1}(z)$ is perpendicular to P_1 and so it is in kernel of the orthogonal projection. $x^2 - \text{proj}_{P_1}(x^2)$ is $x^2 - x + 1/6$ and so is a basis for the 1 dimensional kernel.

(9) (3 points) Find an orthogonal (not necessarily orthonormal) basis for P_2 that extends your orthogonal basis for P_1 from problem (3).

Since the polynomial in problem (8) is perpendicular to P_1 our orthogonal basis is

$$\{1, x - 1/2, x^2 - x + 1/6\}.$$

(10) (3 points) The polynomial of problem (5) minimizes a certain *least squares* integral. What is that integral?

$$ax + b = x - 1/6$$

minimizes

$$\int_0^1 (x^2 - (ax + b))^2 dx.$$

(11) (3 points) If $\det(A) = 5$ and A is an $n \times n$ matrix, then what is $\det(5A)$? (Explain a bit.)

The determinant is linear in each row so we can take a 5 out of each.

$$\det(5A) = 5^n \det(A) = 5^{n+1}.$$

(12) (1 point) If A is an orthogonal rotation $n \times n$ matrix, then what is $\det(5A)$? (Explain a bit.)

An orthogonal rotation matrix has determinant equal to 1, so

$$\det(5A) = 5^n \det(A) = 5^n.$$

(13) (3 points) State Cramer's rule.

See Book.

(14) (3 points) What is the area of the parallelogram defined by the vectors $(1, 5)$ and $(3, 9)$.

The area is the absolute value of the determinant of

$$\begin{pmatrix} 1 & 3 \\ 5 & 9 \end{pmatrix}$$

which is 6.

(15) (3 points) If A is invertible, what is A^{-1} in terms of the adjoint (which you should define of course)?

See book.