110.202. Linear Algebra
2003 Summer
Final
6/16/2003 12:00Noon

1. (20pts) Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find an orthogonal matrix S and a diagonal matrix D such that $S^{-1}AS = D$.
(b) Find a formula for the entries of A^t, where t is a positive integer. Also find the vector $\lim_{t\to\infty} A^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

2. (20pts) Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find a singular value decomposition for A.
(b) Describe the image of the unit circle under the linear transformation $T(\vec{x}) = A\vec{x}$.

3. (10pts) Let q be a quadratic form

$$q(x_1, x_2) = 9x_1^2 - 4x_1x_2 + 6x_2^2.$$

(a) Determine the definiteness of q.
(b) Sketch the curve defined by $q(x_1, x_2) = 1$. Draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

4. (10pts) Decide whether the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is diagonalizable. If possible, find an invertible S and a diagonal D such that $S^{-1}AS = D$.

5. (10pts) Find the trigonometric function of the form

$$f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t)$$

that best fits the data points $(0, -1)$, $\left(\frac{\pi}{2}, 2\right)$, $(\pi, 2)$ and $\left(\frac{3\pi}{2}, 1\right)$, using lease squares.

6. (10pts) Given a matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 7 & 3 & -5 \\ 3 & 2 & 8 & -5 \end{bmatrix}.$$

(a) Find a basis of kernel of A and $\dim(\ker(A))$.
(b) Find a basis of image of A and $\dim(\operatorname{im}(A))$.

7. (10pts) Let T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection in the plane given by the equation

$$x_1 + 2x_2 + 3x_3 = 0.$$

(a) Find the matrix B of this transformation with respect to the basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(b) Use your answer in part (a) to find the standard matrix A of T.

8. (10pts) Consider a linear transformation T from V to W.
(a) For $f_1, f_2, \cdots, f_n \in V$, if $T(f_1), T(f_2), \cdots, T(f_n)$ are linearly independent, show that $f_1, f_2, \cdots, f_n$ are linearly independent.
(b) Assume that $f_1, f_2, \cdots, f_n$ form a basis of V. If T is an isomorphism, show that $T(f_1), T(f_2), \cdots, T(f_n)$ is a basis of W.