110.202 Linear Algebra

Final Solutions

1. (20pts) Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find an orthogonal matrix S and a diagonal matrix D such that $S^{-1}AS = D$.

(b) Find a formula for the entries of A^t, where t is a positive integer. Also find the vector $\lim_{t\to\infty} A^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

[Solution]

(a) Let $f_A(\lambda) = \det(A - \lambda I_3) = \det\left(\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix}\right) = -\lambda^3 + 3\lambda + 2 = -(\lambda - 2)(\lambda + 1)^2 = 0$. We have the eigenvalues are 2 with multiplicity 1 and -1 with multiplicity 2. For $\lambda_1 = 1$, the eigenspace

$$\begin{aligned} E_2 &= \ker\left(\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix}\right) \\ &= \ker\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \\ &= \ker\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

Therefore, $\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ forms an orthonormal basis of E_2.
For $\lambda_2=-1$, the eigenspace

$$\begin{aligned}
E_{-1} &= \ker\left(\begin{bmatrix}0-(-1) & 1 & 1\\ 1 & 0-(-1) & 1\\ 1 & 1 & 0-(-1)\end{bmatrix}\right)\\
&= \ker\left(\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\right)\\
&= \ker\left(\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\right)\\
&= \operatorname{span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}.
\end{aligned}$$

Therefore, $\vec{v}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}$ form a eigenbasis of E_0. Using Gram-schmidt process on $\vec{v}_1$ and $\vec{v}_2$ to get an orthonormal eigenbasis $\vec{w}_1$ and $\vec{w}_2$ for E_0, we have

$$\vec{w}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

and, with $\vec{u}_2=\vec{v}_2-(\vec{w}_1\cdot\vec{v}_2)\,\vec{w}_1=\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}$ and $\|\vec{u}_2\|=\left\|\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\right\|=\frac{\sqrt{6}}{2}$ on hand, we have

$$\begin{aligned}
\vec{w}_2 &= \frac{\vec{v}_2-(\vec{w}_1\cdot\vec{v}_2)\,\vec{w}_1}{\|\vec{v}_2-(\vec{w}_1\cdot\vec{v}_2)\,\vec{w}_1\|}=\frac{\vec{u}_2}{\|\vec{u}_2\|}\\
&= \frac{2}{\sqrt{6}}\left(\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix}\right)=\begin{bmatrix}-\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\end{bmatrix}.
\end{aligned}$$

Set $S = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. We have S is orthogonal and D is diagonal. Therefore, we have $D = S^{-1}AS$ where S is orthogonal and D is diagonal.

(b) From (a), we have $A = SDS^{-1}$. Since S is orthogonal, we have $S^{-1} = S^T$. Hence, for a positive integer t,

$$\begin{aligned}
& A^t \\
= \; & SD^tS^{-1} = SD^tS^T \\
= \; & \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2^t & 0 & 0 \\ 0 & (-1)^t & 0 \\ 0 & 0 & (-1)^t \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \\
= \; & \begin{bmatrix} \frac{1}{3}\left(2^t + 2(-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) \\ \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t + 2(-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) \\ \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t + 2(-1)^t\right) \end{bmatrix}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \lim_{t\to\infty} A^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
= \; & \lim_{t\to\infty} \begin{bmatrix} \frac{1}{3}\left(2^t + 2(-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) \\ \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t + 2(-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) \\ \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t - (-1)^t\right) & \frac{1}{3}\left(2^t + 2(-1)^t\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
= \; & \lim_{t\to\infty} \begin{bmatrix} (-1)^t \\ 0 \\ -(-1)^t \end{bmatrix}.
\end{aligned}$$

That means the limit does not exist since you have two different limit points $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

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2. (20pts) Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find a singular value decomposition for A.

(b) Describe the image of the unit circle under the linear transformation $T(\vec{x}) = A\vec{x}$.

[Solution]

(a) The singular values are the square roots of the eigenvalues of $A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Let $f_{A^TA}(\lambda) = \det\left(A^TA - \lambda I_2\right) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = (\lambda - 3)(\lambda - 1) = 0$. We have the eigenvalues of A^TA are $\lambda_1 = 3$ and $\lambda_2 = 1$. Therefore, the singular values of A are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$$

and

$$\sigma_2 = \sqrt{\lambda_2} = 1.$$

For $\sigma_1 = \sqrt{3}$, the eigenspace $E_3 = \ker\left(\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. Therefore, the nonzero unit vector

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

forms an orthonormal basis of E_3. For $\sigma_2 = 1$, the eigenspace $E_1 = \ker\left(\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$. Therefore, the nonzero unit vector

$$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

forms an orthonormal basis of E_1. Let

$$\begin{aligned} \vec{u}_1 &= \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}
\vec{u}_2 &= \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{1}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.
\end{aligned}$$

Since A is a 3×2 matrix, we know that U will be a 3×3 matrix. So, we need to expand $\{\vec{u}_1, \vec{u}_2\}$ into a orthonormal basis of $\mathbb{R}^3$. That means we need to find a $\vec{u}_3$ such that $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ form an orthonormal basis of $\mathbb{R}^3$. Choose $\vec{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Obviously, $\vec{w}_3$ is not a linear combination of $\vec{u}_1$ and $\vec{u}_2$. So, $\vec{u}_1$, $\vec{u}_2$ and $\vec{w}_3$ form a basis of $\mathbb{R}^3$. Since $\vec{u}_1$ and $\vec{u}_2$ are orthogonal and unit vectors already. We use Gram-Schmidt process on $\vec{w}_3$ to get

$$\vec{u}_3 = \frac{\vec{w}_3 - (\vec{u}_1 \cdot \vec{w}_3)\,\vec{u}_1 - (\vec{u}_2 \cdot \vec{w}_3)\,\vec{u}_2}{\|\vec{w}_3 - (\vec{u}_1 \cdot \vec{w}_3)\,\vec{u}_1 - (\vec{u}_2 \cdot \vec{w}_3)\,\vec{u}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

And, we have $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ form an orthonormal basis of $\mathbb{R}^3$. Let

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore, we get a singular value decomposition(SVD) for $A = U\Sigma V^T$.

(b) From (a), we have

$$\begin{aligned} T(\vec{v}_1) &= A\vec{v}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \sqrt{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} T(\vec{v}_2) &= A\vec{v}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}. \end{aligned}$$

The unit circle in $\mathbb{R}^2$ consists of all vectors of the form

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \text{ where } c_1^2 + c_2^2 = 1.$$

Therefore, the image of the unit circle under T consists of the vectors

$$T(\vec{x}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) \text{ where } c_1^2 + c_2^2 = 1.$$

That means

$$T(\vec{x}) = c_1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \sqrt{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ where } c_1^2 + c_2^2 = 1.$$

This is an ellipse with the semimajor axe is the line generated by $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \sqrt{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ with the length $\sigma_1 = \sqrt{3}$ and the semiminor axe is the line generated by $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ with the length $\sigma_1 = 1$.

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3. (10pts) Let q be a quadratic form

$$q(x_1, x_2) = 9x_1^2 - 4x_1x_2 + 6x_2^2.$$

(a) Determine the definiteness of q.

(b) Sketch the curve defined by $q(x_1, x_2) = 1$. Draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

[Solution]

(a) Let $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$. We have $q(\vec{x}) = \vec{x} \cdot A\vec{x}$.
[Method 1]
Calculate

$$A^{(1)} = \det([9]) = 9 > 0$$

and

$$A^{(2)} = \det\left(\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\right) = 50 > 0.$$

By the theorem in the textbook, we have q is positive definite.
[Method 2]
Let λ_1 and λ_2 be the two eigenvalues of A with associated eigenvectors $\vec{v}_1$ and $\vec{v}_2$, respectively. We have $\lambda_1\lambda_2 = \det(A) = 54 - 4 = 50 > 0$ and $\lambda_1 + \lambda_2 = \operatorname{tr}(A) = 9 + 6 = 15 > 0$. This implies that $\lambda_1 > 0$ and $\lambda_2 > 0$. Moreover, we have

$$q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$$

where $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. That means $q(\vec{x}) > 0$ for all $\vec{x} \neq 0$. By the definition of definiteness, A is positive definite.

(b) Let $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$. We have $q(\vec{x}) = \vec{x} \cdot A\vec{x}$. Set $0 = f_A(\lambda) = \det(A - \lambda I_2) = \det\left(\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right) = \lambda^2 - 15\lambda + 50 = (\lambda - 10)(\lambda - 5)$. We get the eigenvalues $\lambda_1 = 10$ and $\lambda_2 = 5$. For $\lambda_1 = 10$, the eigenspace

$$\begin{aligned} E_{10} &= \ker\left(\begin{bmatrix} 9-10 & -2 \\ -2 & 6-10 \end{bmatrix}\right) \\ &= \ker\left(\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}. \end{aligned}$$

Therefore, $\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ forms an orthonormal basis of E_{10}.
For $\lambda_2 = 5$, the eigenspace

$$\begin{aligned} E_5 &= \ker\left(\begin{bmatrix} 9-5 & -2 \\ -2 & 6-5 \end{bmatrix} \right) \\ &= \ker\left(\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \right) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}. \end{aligned}$$

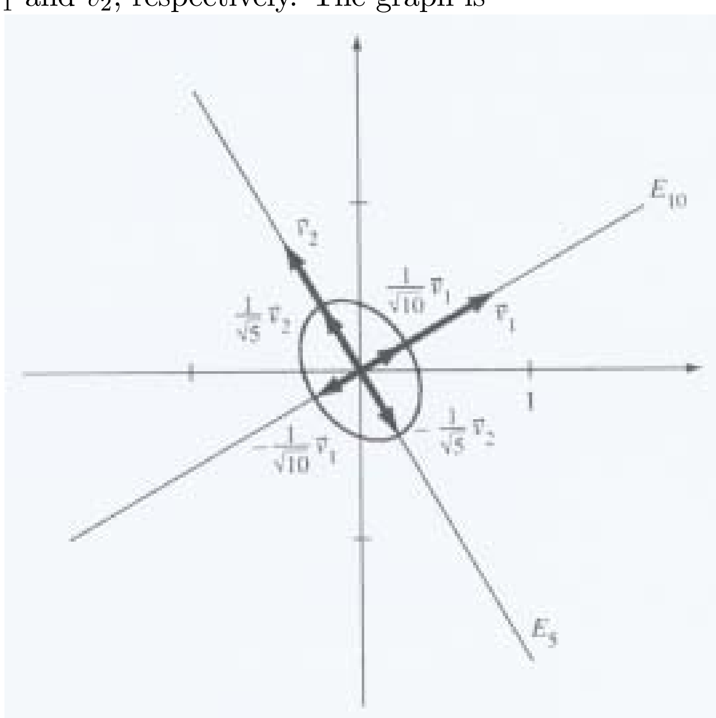
Therefore, $\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ forms an orthonormal basis of E_5.
Hence we have an orthonormal eigenbasis

$$\vec{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Moreover, the curve can be re-written as

$$10c_1^2 + 5c_2^2 = 1.$$

That means the curve is a ellipse in $c_1 - c_2$ coordinates system where the principal axes are E_{10} and E_5 which are generated by $\vec{v}_1$ and $\vec{v}_2$, respectively. The graph is



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4. (10pts) Decide whether the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is diagonalizable. If possible, find an invertible S and a diagonal D such that $S^{-1}AS = D$.

[Solution]

Using $f_A(\lambda) = \det(A - \lambda I_3) = \det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 0-\lambda \end{bmatrix}\right) = -(1-\lambda)^2\lambda = 0$, we have eigenvalues are 1, 1 and 0. For $\lambda_1 = 1$, the eigenspace

$$\begin{aligned} E_1 &= \ker(A - 1\cdot I_3) = \ker\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}\right) \\ &= \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}. \end{aligned}$$

For $\lambda_2 = 0$, the eigenspace

$$\begin{aligned} E_0 &= \ker(A - 0\cdot I_3) = \ker\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) \\ &= \ker\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

Since $\dim(E_1) + \dim(E_0) = 1 + 1 = 2 < 3$, we have no eigenbasis of $\mathbb{R}^3$ in this case. This implies that A is not diagonalizable.
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5. (10pts) Find the trigonometric function of the form

$$f(t) = c_0 + c_1\sin(t) + c_2\cos(t)$$

that best fits the data points $(0,-1)$, $\left(\frac{\pi}{2}, 2\right)$, $(\pi, 2)$ and $\left(\frac{3\pi}{2}, 1\right)$, using lease squares.

[Solution]

We want to find a $f(t) = c_0 + c_1\sin(t) + c_2\cos(t)$ such that $f(0) = -1$, $f\left(\frac{\pi}{2}\right) = 2$, $f(\pi) = 2$ and $f\left(\frac{3\pi}{2}\right) = 1$. These conditions give the system of linear equations

$$\begin{cases} c_0 & + & c_1 \sin(0) & + & c_2 \cos(0) & = & f(0) & = & -1 \\ c_0 & + & c_1 \sin\left(\frac{\pi}{2}\right) & + & c_2 \cos\left(\frac{\pi}{2}\right) & = & f\left(\frac{\pi}{2}\right) & = & 2 \\ c_0 & + & c_1 \sin(\pi) & + & c_2 \cos(\pi) & = & f(\pi) & = & 2 \\ c_0 & + & c_1 \sin\left(\frac{3\pi}{2}\right) & + & c_2 \cos\left(\frac{3\pi}{2}\right) & = & f\left(\frac{3\pi}{2}\right) & = & 1 \end{cases},$$

or,

$$\begin{cases} c_0 & & & + & c_2 & = & -1 \\ c_0 & + & c_1 & & & = & 2 \\ c_0 & & & - & c_2 & = & 2 \\ c_0 & - & c_1 & & & = & 1 \end{cases}.$$

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$. We can write the system as $A\vec{x} = \vec{b}$. Since $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we have $\ker(A) = \{0\}$. So, the unique least-squares solution of $A\vec{x} = \vec{b}$ is

$$\begin{aligned} \vec{x}^* &= \left(A^T A\right)^{-1} A^T \vec{b} \\ &= \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}. \end{aligned}$$

Moreover, the trigonometric function,

$$f^*(t) = 1 + \frac{1}{2}\sin(t) - \frac{3}{2}\cos(t),$$

best fits the data points $(0, -1)$, $\left(\frac{\pi}{2}, 2\right)$, $(\pi, 2)$ and $\left(\frac{3\pi}{2}, 1\right)$ in the least-squares sense.

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6. (10pts) Given a matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 7 & 3 & -5 \\ 3 & 2 & 8 & -5 \end{bmatrix}.$$

(a) Find a basis of kernel of A and $\dim(\ker(A))$.
(b) Find a basis of image of A and $\dim(\operatorname{im}(A))$.

[Solution]
By Gauss-Jordan elimination, we have

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 7 & 3 & -5 \\ 3 & 2 & 8 & -5 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 7 & 7 & -7 \\ 0 & 2 & 2 & -2 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(A).$$

(a) Assume $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \ker A$. Then we have $A\vec{x} = \vec{0}$. By Gauss-Jordan elimination(which implies $\ker(A) = \ker \operatorname{rref}(A)$), we have

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}.$$

Assume $x_3 = s$ and $x_4 = t$ for all $s, t \in \mathbb{R}$. We have solutions of the system,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + t \\ -s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Assume $\vec{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. That means $\ker(A) \subseteq \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$. Moreover, we check that

$$A(\vec{v}_1) = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 7 & 3 & -5 \\ 3 & 2 & 8 & -5 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$A(\vec{v}_2) = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -2 & 7 & 3 & -5 \\ 3 & 2 & 8 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This tells us that $\{\vec{v}_1, \vec{v}_2\} \subseteq \ker(A)$ which implies $\ker(A) = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$. To check $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, we assume that $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$, or

$$0 = c_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2 \\ -c_1 + c_2 \\ c_1 \\ c_2 \end{bmatrix}.$$

This implies that $c_1 = c_2 = 0$. That proves that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. Now, we can say $\vec{v}_1, \vec{v}_2$ form a basis of $\ker(A)$. And, $\dim(\ker(A))$ equals the number of vectors in a basis. So, $\dim(\ker(A)) = 2$.

(b) Set $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be the column vectors of A. Set $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$ be the column vectors of rref A. We know that

$$\operatorname{im}(A) = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}.$$

From our reduced row-echlon form, we can read two relations of $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$,

$$\vec{w}_3 = 2\vec{w}_1 + \vec{w}_2$$

and

$$\vec{w}_4 = -\vec{w}_1 - \vec{w}_2.$$

This implies we have the same relation of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ which are

$$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$$

and

$$\vec{v}_4 = -\vec{v}_1 - \vec{v}_2.$$

Therefore, we can conclude that $\vec{v}_1, \vec{v}_2$ are linearly independent and

$$\operatorname{im}(A) = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}.$$

Now, we can say that $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$ form a basis of $\operatorname{im}(A)$. And, $\dim(\operatorname{im}(A))$ equals the number of vectors in a basis. So, $\dim(\operatorname{im}(A)) = 2$.

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7. (10pts) Let T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection in the plane given by the equation

$$x_1 + 2x_2 + 3x_3 = 0.$$

(a) Find the matrix B of this transformation with respect to the basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(b) Use your answer in part (a) to find the standard matrix A of T.

[**Solution**]

(a) Let P be the plane given by the equation $x_1 + 2x_2 + 3x_3 = 0$. We observe that $\vec{v}_1$ and $\vec{v}_2$ are both on the plane P. Since T is a reflection, it keeps $\vec{v}_1$ and $\vec{v}_2$ unchanged, that is, $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = \vec{v}_2$. And, $\vec{v}_3$ is the normal vector of P. That means $\vec{v}_3$ is perpendicular to the plane P. Therefore, since T is a reflection in P, we have $T(\vec{v}_3) = -\vec{v}_3$. With respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, we have $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}$, $[T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}}$ and $[T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}_{\mathcal{B}}$. So

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & [T(\vec{v}_2)]_{\mathcal{B}} & [T(\vec{v}_3)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) Write $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}$. By the theorem in the textbook, we have $A = SBS^{-1}$. To find S^{-1}, we use Gauss-Jordan elimination on

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$\longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 1 \end{array}\right]$$

$\longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{14}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$\longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{14} & \frac{2}{14} & -\frac{4}{14} \\ 0 & 1 & 0 & -\frac{5}{14} & \frac{4}{14} & -\frac{1}{14} \\ 0 & 0 & 1 & \frac{1}{14} & \frac{2}{14} & \frac{3}{14} \end{array}\right].$$

Hence,

$$S^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{14} & \frac{2}{7} & -\frac{1}{14} \\ \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \end{bmatrix}$$

and

$$A = SBS^{-1} = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix}.$$

■

8. (10pts) Consider a linear transformation T from V to W.

(a) For $f_1, f_2, \cdots, f_n \in V$, if $T(f_1), T(f_2), \cdots, T(f_n)$ are linearly independent, show that $f_1, f_2, \cdots, f_n$ are linearly independent.

(b) Assume that $f_1, f_2, \cdots, f_n$ form a basis of V. If T is an isomorphism, show that $T(f_1), T(f_2), \cdots, T(f_n)$ is a basis of W.

[Solution]

(a) Assume

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = \vec{0}.$$

By applying T on both sides, we have

$$T(a_1 f_1 + a_2 f_2 + \cdots + a_n f_n) = T\left(\vec{0}\right) = \vec{0}.$$

Since T is a linear transformation,

$$a_1 T(f_1) + \cdots + a_n T(f_n) = T(a_1 f_1 + \cdots + a_n f_n) = \vec{0}.$$

Since $T(f_1), T(f_2), \cdots, T(f_n)$ are linearly independent, we have $a_1 = a_2 = \cdots = a_n = 0$. That tells us that $f_1, f_2, \cdots, f_n$ are linearly independent.

(b) To show that $T(f_1), T(f_2), \cdots, T(f_n)$ are linearly independent, we assume $c_1 T(f_1) + c_2 T(f_2) + \cdots + c_n T(f_n) = 0$. Hence, we have $T(c_1 f_1 + \cdots + c_n f_n) = 0$ since T is a linear transformation. Since T is an isomorphism from V to W, we have T is invertible, that means, $\ker(T) = \{0\}$. This implies $c_1 f_1 + \cdots + c_n f_n = 0$. Moreover, $f_1, f_2, \cdots, f_n$ form a basis of V. This condition forces that $c_1 = c_2 = \cdots = c_n = 0$. So, $T(f_1), T(f_2), \cdots, T(f_n)$ are linearly independent. Since T is an isomorphism from V to W, we have T^{-1} is an isomorphism from W to V such that $T(T^{-1}(g)) = g$ for all $g \in W$. For all $g \in W$, we have $T^{-1}(g) \in V$. Since $f_1, f_2, \cdots, f_n$ form a basis of V, there exist $t_1, t_2, \cdots, t_n$ such that $T^{-1}(g) = t_1 f_1 + \cdots + t_n f_n$. Therefore, $g = T(T^{-1}(g)) = T(t_1 f_1 + \cdots + t_n f_n) = t_1 T(f_1) + \cdots t_n T(f_n)$. That means g is a linear combination of $T(f_1), T(f_2), \cdots, T(f_n)$. And, obviously, $T(f_1), T(f_2), \cdots, T(f_n)$ are all in W. Thus, we can conclude that $W = \operatorname{span}\{T(f_1), T(f_2), \cdots, T(f_n)\}$. So, $T(f_1), T(f_2), \cdots, T(f_n)$ form a basis of W

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