110.202. Linear Algebra
2003 Summer
Midterm I
6/5/2003 1:00PM

1. (10pts) Consider a matrix A, and let $B = \text{rref}\,(A)$.
(a) Is $\ker(A)$ necessarily equal to $\ker(B)$? Explain.
(b) Is $\text{im}\,(A)$ necessarily equal to $\text{im}\,(B)$? Explain.

2. (15pts) Consider the $n \times n$ matrix M_n which contains all integers $1, 2, 3, \cdots, n^2$ as its entries, written in sequence, column by column; for example,

$$M_4 = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}.$$

(a) Determine the rank of M_4.
(b) Determine the rank of M_n, for an arbitrary $n \geq 2$.
(c) For which integers n is M_n invertible?

3. (15pts) If A and B are two $n \times n$ matrices such that $BA = I_n$. Prove the following properties:
(a) A and B are both invertible.
(b) $A^{-1} = B$ and $B^{-1} = A$.
(c) $AB = I_n$.

4. (20pts) Let T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection in the plane given by the equation

$$x_1 + 2x_2 + 3x_3 = 0.$$

(a) Find the matrix B of this transformation with respect to the basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(b) Use your answer in part (a) to find the standard matrix A of T.

5. (10pts) Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(a) Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ be vectors in $\mathbb{R}^n$. If $T(\vec{v}_1), T(\vec{v}_2), \cdots, T(\vec{v}_q)$ are linearly independent, are $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ linearly independent? How can you tell?

(b) Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ be vectors in $\mathbb{R}^n$. If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ are linearly independent, are $T(\vec{v}_1), T(\vec{v}_2), \cdots, T(\vec{v}_q)$ linearly independent? How can you tell?

6. (20pts) Given a matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -5 & 0 \\ -2 & 7 & 3 & 4 & 0 \\ 3 & 2 & 8 & 1 & -4 \\ 4 & -1 & 8 & 2 & -9 \end{bmatrix}.$$

(a) Find a basis of kernel of A and $\dim(\ker(A))$.

(b) Find a basis of image of A and $\dim(\operatorname{im}(A))$.

7. (10pts) Let L be a line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

(b) Find a matrix A such that $\operatorname{proj}_L(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.