110.202 Linear Algebra

Midterm 1 Solutions

1. (10pts) Consider a matrix A, and let $B = \text{rref}(A)$.

(a) Is $\ker(A)$ necessarily equal to $\ker(B)$? Explain.

(b) Is $\text{im}(A)$ necessarily equal to $\text{im}(B)$? Explain.

[Solution]

(a) Yes. By construction of the reduced row-echlon form, the system $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions(the whole process of Gauss-Jordan elimination doesn't change the solutions of a system).

(b) No. Choose $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then, we have $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence,

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \text{ where } y \in \mathbb{R} \right\}$$

and

$$\text{im}(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ where } x \in \mathbb{R} \right\}$$

which tell us that $\text{im}(A) \neq \text{im}(B)$.

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2. (15pts) Consider the $n \times n$ matrix M_n which contains all integers $1, 2, 3, \cdots, n^2$ as its entries, written in sequence, column by column; for example,

$$M_4 = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}.$$

(a) Determine the rank of M_4.

(b) Determine the rank of M_n, for an arbitrary $n \geq 2$.

(c) For which integers n is M_n invertible?

[Solution]

(a) By Gauss-Jordan elimination, we have

$$M_4 = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 5 & 9 & 13 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, the rank $(M_4) = 2$.

(b) By Gauss-Jordan elimination, we have

$$M_n = \begin{bmatrix} 1 & n+1 & 2n+1 & \cdots & (n-1)\,n+1 \\ 2 & n+2 & 2n+2 & \cdots & (n-1)\,n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 2n & 3n & \cdots & n^2 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & n+1 & 2n+1 & \cdots & (n-1)\,n+1 \\ 0 & n+2-2\,(n+1) & 2n+2-2\,(2n+1) & \cdots & (n-1)\,n+2-2\,((n-1)\,n+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 2n-n\,(n+1) & 3n-n\,(2n+1) & \cdots & n^2-n\,((n-1)\,n+1) \end{bmatrix}$$

$\longrightarrow$(by using the fact: $sn + t - t\,(sn+1) = sn - tsn = -\,(t-1)\,sn$ for all $1 \le s \le n-1$ and $2 \le t \le n$)

$$\begin{bmatrix} 1 & n+1 & 2n+1 & \cdots & (n-1)\,n+1 \\ 0 & -n & -2n & \cdots & -\,(n-1)\,n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\,(n-1)\,n & -\,(n-1)\,2n & \cdots & -\,(n-1)\,(n-1)\,n \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & -1 & \cdots & -(n-2) \\ 0 & 1 & 2 & \cdots & (n-1) \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, the rank $(M_n) = 2$.

(c) Since M_n is invertible if and only if rank $M_n = n$. By part (b), we have M_1 and M_2 with are invertible. M_n for $n \geq 3$ is not invertible.

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3. (15pts) If A and B are two $n \times n$ matrices such that $BA = I_n$. Prove the following properties:

(a) A and B are both invertible.

(b) $A^{-1} = B$ and $B^{-1} = A$.

(c) $AB = I_n$.

[Solution]

1. (a) To prove A is invertible, we need to show that $A\vec{x} = \vec{0}$ has exact one solution. Assume $\vec{y}$ be a solution of $A\vec{x} = \vec{0}$. Hence, we have $A\vec{y} = \vec{0}$. This implies

$$\vec{y} = I_n\vec{y} = (BA)\,\vec{y} = B\,(A\vec{y}) = \vec{0}.$$

Therefore, $\vec{0}$ is the only solution of $A\vec{x} = \vec{0}$. This tells us that A is invertible. We will prove (b) first and use (b) to prove B is invertible.

(b) Since A is invertible, there exists a A^{-1} such that $A^{-1}A = I_n = AA^{-1}$. And, we have

$$A^{-1} = I_nA^{-1} = (BA)\,A^{-1} = B\left(AA^{-1}\right) = BI_n = B.$$

By the definition of inverse linear transformation, we have if a linear transformation T is invertible, then so is T^{-1} and $\left(T^{-1}\right)^{-1} = T$. Assume $T(\vec{x}) = A\vec{x}$. we have $T^{-1}(\vec{y}) = A^{-1}\vec{y} = B\vec{y}$. This implies B is invertible and

$$B^{-1} = \left(A^{-1}\right)^{-1} = A.$$

This completes the proof of (a) and (b).

(c) Since $A^{-1} = B$, we have $AB = AA^{-1} = I_n$.

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4. (20pts) Let T from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection in the plane given by the equation

$$x_1 + 2x_2 + 3x_3 = 0.$$

(a) Find the matrix B of this transformation with respect to the basis

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(b) Use your answer in part (a) to find the standard matrix A of T.

[Solution]

(a) Let P be the plane given by the equation $x_1 + 2x_2 + 3x_3 = 0$. We observe that $\vec{v}_1$ and $\vec{v}_2$ are both on the plane P. Since T is a reflection, it keeps $\vec{v}_1$ and $\vec{v}_2$ unchanged, that is, $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = \vec{v}_2$. And, $\vec{v}_3$ is the normal vector of P. That means $\vec{v}_3$ is perpendicular to the plane P. Therefore, since T is a reflection in P, we have $T(\vec{v}_3) = -\vec{v}_3$. With respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, we have $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}$,

$[T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}}$ and $[T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}_{\mathcal{B}}$. So

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & [T(\vec{v}_2)]_{\mathcal{B}} & [T(\vec{v}_3)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) Write $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}$. By the theorem in the textbook, we have $A = SBS^{-1}$. To find S^{-1}, we use Gauss-Jordan elimination on

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$\longrightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{14}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$$\longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{14} & \frac{2}{14} & -\frac{4}{14} \\ 0 & 1 & 0 & -\frac{5}{14} & \frac{4}{14} & -\frac{1}{14} \\ 0 & 0 & 1 & \frac{1}{14} & \frac{2}{14} & \frac{3}{14} \end{array}\right].$$

Hence,

$$S^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{14} & \frac{2}{7} & -\frac{1}{14} \\ \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \end{bmatrix}$$

and

$$A = SBS^{-1} = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix}.$$

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5. (10pts) Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(a) Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ be vectors in $\mathbb{R}^n$. If $T(\vec{v}_1), T(\vec{v}_2), \cdots, T(\vec{v}_q)$ are linearly independent, are $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ linearly independent? How can you tell?

(b) Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ be vectors in $\mathbb{R}^n$. If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ are linearly independent, are $T(\vec{v}_1), T(\vec{v}_2), \cdots, T(\vec{v}_q)$ linearly independent? How can you tell?

[**Solution**]

(a) Yes! Assume

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_q\vec{v}_q = \vec{0}.$$

By applying T on both sides, we have

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_q\vec{v}_q) = T\left(\vec{0}\right) = \vec{0}.$$

Since T is a linear transformation,

$$a_1T(\vec{v}_1) + \cdots + a_qT(\vec{v}_q) = T(a_1\vec{v}_1 + \cdots + a_q\vec{v}_q) = \vec{0}.$$

Since $T(\vec{v}_1), T(\vec{v}_2), \cdots, T(\vec{v}_q)$ are linearly independent, we have $a_1 = a_2 = \cdots = a_m = 0$. That tells us that $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q$ are linearly independent.

(b) No! Let $T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. $\vec{e}_1$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent in $\mathbb{R}^2$. But, $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ are not linearly independent since we have $T(\vec{e}_1) - T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 0$.

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6. (20pts) Given a matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -5 & 0 \\ -2 & 7 & 3 & 4 & 0 \\ 3 & 2 & 8 & 1 & -4 \\ 4 & -1 & 8 & 2 & -9 \end{bmatrix}.$$

(a) Find a basis of kernel of A and $\dim(\ker(A))$.
(b) Find a basis of image of A and $\dim(\operatorname{im}(A))$.

[Solution]
By Gauss-Jordan elimination, we have

$$A = \begin{bmatrix} 1 & 0 & 2 & -5 & 0 \\ -2 & 7 & 3 & 4 & 0 \\ 3 & 2 & 8 & 1 & -4 \\ 4 & -1 & 8 & 2 & -9 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 2 & -5 & 0 \\ 0 & 7 & 7 & -6 & 0 \\ 0 & 2 & 2 & 16 & -4 \\ 0 & -1 & 0 & 22 & -9 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 2 & -5 & 0 \\ 0 & 1 & 0 & -22 & 9 \\ 0 & 2 & 2 & 16 & -4 \\ 0 & 7 & 7 & -6 & 0 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 2 & -5 & 0 \\ 0 & 1 & 0 & -22 & 9 \\ 0 & 0 & 2 & 60 & -22 \\ 0 & 0 & 7 & 148 & -63 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & -65 & 22 \\ 0 & 1 & 0 & -22 & 9 \\ 0 & 0 & 1 & 30 & -11 \\ 0 & 0 & 0 & -62 & 14 \end{bmatrix}$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{227}{31} \\ 0 & 1 & 0 & 0 & \frac{125}{31} \\ 0 & 0 & 1 & 0 & -\frac{131}{31} \\ 0 & 0 & 0 & 1 & -\frac{7}{31} \end{bmatrix} = \operatorname{rref}(A).$$

(a) Assume $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \ker A$. Then we have $A\vec{x} = \vec{0}$. By Gauss-Jordan elimination(which implies $\ker(A) = \ker \operatorname{rref}(A)$), we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{227}{31} \\ 0 & 1 & 0 & 0 & \frac{125}{31} \\ 0 & 0 & 1 & 0 & -\frac{131}{31} \\ 0 & 0 & 0 & 1 & -\frac{7}{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{0}.$$

Assume $x_5 = t$ for all $s, t \in \mathbb{R}$. We have solutions of the system,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{227}{31}t \\ -\frac{125}{31}t \\ \frac{131}{31}t \\ \frac{7}{31}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{227}{31} \\ -\frac{125}{31} \\ \frac{131}{31} \\ \frac{7}{31} \\ 1 \end{bmatrix} = \frac{t}{31} \begin{bmatrix} -227 \\ -125 \\ 131 \\ 7 \\ 31 \end{bmatrix}.$$

Assume $\vec{v} = \begin{bmatrix} -227 \\ -125 \\ 131 \\ 7 \\ 31 \end{bmatrix}$. That means $\ker(A) \subseteq \operatorname{span}\{\vec{v}\}$.
Moreover, for all $k \in \mathbb{R}$,

$$A(k\vec{v}) = k \begin{bmatrix} 1 & 0 & 2 & -5 & 0 \\ -2 & 7 & 3 & 4 & 0 \\ 3 & 2 & 8 & 1 & -4 \\ 4 & -1 & 8 & 2 & -9 \end{bmatrix} \begin{bmatrix} -227 \\ -125 \\ 131 \\ 7 \\ 31 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This tells us that which $\operatorname{span}\{\vec{v}\} \subseteq \ker(A)$ which implies $\ker(A) = \operatorname{span}\{\vec{v}\}$. Since $\vec{v}$ is a nonzero vector, $\vec{v}$ is linearly independent. Now, we can say $\{\vec{v}\}$ is a basis of $\ker(A)$. And, $\dim(\ker(A))$ equals the number of vectors in a basis. So, $\dim(\ker(A)) = 1$.

(b) Set $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$ to be column vectors of A. Set $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{w}_5$ to be column vectors of rref A. We know that

$$\operatorname{im}(A) = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}.$$

From our reduced row-echlon form, we can read there is only one relation of $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{w}_5$,

$$\frac{227}{31}\vec{w}_1 + \frac{125}{31}\vec{w}_2 - \frac{131}{31}\vec{w}_3 - \frac{7}{31}\vec{w}_4 = \vec{w}_5.$$

This implies we have the same relation of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$,

$$\frac{227}{31}\vec{v}_1 + \frac{125}{31}\vec{v}_2 - \frac{131}{31}\vec{v}_3 - \frac{7}{31}\vec{v}_4 = \vec{v}_5.$$

And, that is the only relation of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$. Then we can conclude that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent and

$$\operatorname{im}(A) = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}.$$

Now, we can say that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis of $\operatorname{im}(A)$. And, $\dim(\operatorname{im}(A))$ equals the number of vectors in a basis. So, $\dim(\operatorname{im}(A)) = 4$.

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7. (10pts) Let L be a line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

(b) Find a matrix A such that $\mathrm{proj}_L(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^3$.

[**Solution**]

(a) Since $\left\| \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\| = \sqrt{2^2 + 1^2 + 2^2} = 3$, we have a unit vector of L,

$$\vec{u} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

By the formula of projection, we have

$$\mathrm{proj}_L(\vec{x}) = (\vec{u} \cdot \vec{x})\,\vec{u}.$$

Hence the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L is

$$\begin{aligned} \mathrm{proj}_L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) &= \left(\frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 1\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \\ &= \left(\frac{5}{3}\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}. \end{aligned}$$

(b) To calculate the matrix A of the orthogonal projection onto L, $\mathrm{proj}_L(-)$, we calculate $\mathrm{proj}_L(\vec{e}_1)$, $\mathrm{proj}_L(\vec{e}_2)$ and $\mathrm{proj}_L(\vec{e}_3)$ first. We have

$$\begin{aligned} \mathrm{proj}_L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) &= \left(\frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 0\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \\ &= \left(\frac{2}{3}\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ \frac{2}{9} \\ \frac{4}{9} \end{bmatrix}. \end{aligned}$$

And,

$$\begin{aligned}
\mathrm{proj}_L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) &= \left(\frac{2}{3}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 0\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \\
&= \left(\frac{1}{3}\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{2}{9} \end{bmatrix}.
\end{aligned}$$

And,

$$\begin{aligned}
\mathrm{proj}_L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) &= \left(\frac{2}{3}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 1\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \\
&= \left(\frac{2}{3}\right)\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ \frac{2}{9} \\ \frac{4}{9} \end{bmatrix}.
\end{aligned}$$

So, the matrix

$$A = \begin{bmatrix} \mathrm{proj}_L(\vec{e}_1) & \mathrm{proj}_L(\vec{e}_2) & \mathrm{proj}_L(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}.$$

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