110.202. Linear Algebra
2003 Summer
Midterm II
6/17/2003 1:00PM

1. (20pts) (a) Find an orthonormal basis of the space \( P_1 \) with inner product

\[
\langle f, g \rangle = \int_0^1 f(t)g(t)dt.
\]

(b) Find the linear polynomial \( g(t) = a + bt \) that best approximates the function \( f(t) = t^2 \) in the interval \([0,1]\) in the (continuous) least-squares sense.

2. (10pts) Consider the subspace \( W \) of \( \mathbb{R}^4 \) spanned by the vectors

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}.
\]

Find the matrix of the orthogonal projection onto \( W \).

3. (10pts) Use Cramer’s rule to solve the system

\[
\begin{align*}
x_1 + x_3 &= 1 \\
2x_1 - 4x_2 + 5x_3 &= 0 \\
-2x_2 - x_3 &= 4
\end{align*}
\]

4. (10pts) Find the determinant of

\[
A = \begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 0 & 3 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & -1 & 7 & 2 \end{bmatrix}.
\]

5. (10pts) Find the trigonometric function of the form

\[
f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t)
\]

that best fits the data points \((0, -1), \left(\frac{\pi}{2}, 2\right), (\pi, 2)\) and \((\frac{3\pi}{2}, 1)\), using least squares.
6. (20pts) Find the \(QR\) factorization of
\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & -1 & -2 \\
-1 & 1 & 0
\end{bmatrix}.
\]

7. (10pts) Find the matrix of the linear transformation,
\[
T(M) = \begin{bmatrix}
0 & 1 \\
0 & -1
\end{bmatrix} M
\]
from \(M_{2 \times 2}\) to \(M_{2 \times 2}\) with respect to the basis,
\[
\begin{bmatrix}
1 & 0 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
2 & 1 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
1 & 2
\end{bmatrix}, \begin{bmatrix}
0 & -1 \\
0 & 1
\end{bmatrix}
\]
and determine whether \(T\) is an isomorphism.

8. (10pts) Let \(V\) and \(W\) be linear spaces. Let \(T\) be an isomorphism from \(V\) to \(W\). Assume that \(f_1, f_2, \ldots, f_n\) form a basis of \(V\). Show that \(T(f_1), T(f_2), \ldots, T(f_n)\) is a basis of \(W\).