110.202 Linear Algebra

Midterm 2 Solutions

1. (20pts) (a) Find an orthonormal basis of the space P_1 with inner product

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt.$$

(b) Find the linear polynomial $g(t) = a + bt$ that best approximates the function $f(t) = t^2$ in the interval $[0, 1]$ in the (continuous) least-squares sense.

[Solution]

(a) In P_1, we have a standard basis $\{1, t\}$. By using Grad-Schmidt process with inner product $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$, we have

$$g_1(t) = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_0^1 1 \cdot 1 dt}} = 1$$

and

$$\begin{aligned} g_2(t) &= \frac{t - \langle 1, t \rangle 1}{\|t - \langle 1, t \rangle 1\|} = \frac{t - \int_0^1 t dt}{\left\| t - \int_0^1 t dt \right\|} = \frac{t - \frac{1}{2}}{\left\| t - \frac{1}{2} \right\|} \\ &= \frac{t - \frac{1}{2}}{\sqrt{\int_0^1 \left(t - \frac{1}{2}\right)^2 dt}} = \sqrt{3}\,(2t - 1). \end{aligned}$$

Therefore, $g_1(t)$ and $g_2(t)$ form a orthonormal basis of P_1 with inner product $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$.

(b) By Fact 5.5.3 in the textbook, to find a linear polynomial $g(t) = a + bt$ that best approximates the function $f(t) = t^2$ in the interval $[0, 1]$ in the (continuous) least-squares sense, we are looking for the projection of t^2 onto P_1, $\operatorname{proj}_{P_1} t^2$. With respect to the inner product $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$, we have

$$\begin{aligned}
\operatorname{proj}_{P_1} t^2 &= \left\langle g_1(t), t^2 \right\rangle g_1(t) + \left\langle g_2(t), t^2 \right\rangle g_2(t) \\
&= \left(\int_0^1 1 \cdot t^2 dt \right) 1 \\
&\quad + \left(\int_0^1 \left(\sqrt{3}\,(2t-1) \right) t^2 dt \right) \sqrt{3}\,(2t-1) \\
&= \frac{1}{3} + \left(\frac{\sqrt{3}}{6} \right) \sqrt{3}\,(2t-1) \\
&= t - \frac{1}{6}.
\end{aligned}$$

So, $g(t) = t - \frac{1}{6}$ is the best approximation of the function $f(t) = t^2$ in the interval $[0, 1]$ in the (continuous) least-squares sense.

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2. (10pts) Consider the subspace W of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}.$$

Find the matrix of the orthogonal projection onto W.

[Solution]

To get an orthonormal basis of W, we use Gram-Schmidt process for $\vec{v}_1$ and $\vec{v}_2$. So, we have

$$\vec{w}_1 = \frac{\vec{v}_1}{\|v_1\|} = \frac{1}{\sqrt{1^2 + 1^2 + 1^2 + 1^2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

and

$$\vec{w}_2 = \frac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\, \vec{w}_1}{\|\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\, \vec{w}_1\|} = \begin{bmatrix} -\frac{1}{10} \\ \frac{7}{10} \\ -\frac{7}{10} \\ \frac{1}{10} \end{bmatrix}.$$

Set

$$A = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{10} \\ \frac{1}{2} & \frac{7}{10} \\ \frac{1}{2} & -\frac{7}{10} \\ \frac{1}{2} & \frac{1}{10} \end{bmatrix}.$$

By theorem derived on the textbook, we have the matrix of the projection onto W is AA^T,

$$AA^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{10} \\ \frac{1}{2} & \frac{7}{10} \\ \frac{1}{2} & -\frac{7}{10} \\ \frac{1}{2} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{10} & \frac{7}{10} & -\frac{7}{10} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{13}{50} & \frac{9}{50} & \frac{8}{25} & \frac{6}{25} \\ \frac{9}{50} & \frac{37}{50} & -\frac{6}{25} & \frac{8}{25} \\ \frac{8}{25} & -\frac{6}{25} & \frac{37}{50} & \frac{9}{50} \\ \frac{6}{25} & \frac{8}{25} & \frac{9}{50} & \frac{13}{50} \end{bmatrix}.$$

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3. (10pts) Use Cramer's rule to solve the system

$$\begin{cases} x_1 & & + & x_3 & = & 1 \\ 2x_1 & - \; 4x_2 & + & 5x_3 & = & 0 \\ & - \; 2x_2 & - & x_3 & = & 4 \end{cases}.$$

[Solution]

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 5 \\ 0 & -2 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$.We have $\det(A) = 10 \neq 0$. Therefore, we can use Cramer's rule to get the solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for the system $A\vec{x} = \vec{b}$. Let $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 5 \\ 4 & -2 & -1 \end{bmatrix}$ and $\det(A_1) = 30$. Let $A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 5 \\ 0 & 4 & -1 \end{bmatrix}$ and $\det(A_2) = -10$. Let $A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 0 \\ 0 & -2 & 4 \end{bmatrix}$ and $\det(A_3) = -20$. By Cramer's rule, we have

$$\begin{aligned} x_1 &= \frac{\det(A_1)}{\det(A)} = \frac{30}{10} = 3, \\ x_2 &= \frac{\det(A_2)}{\det(A)} = \frac{-10}{10} = -1, \\ x_3 &= \frac{\det(A_3)}{\det(A)} = \frac{-20}{10} = -2. \end{aligned}$$

Therefore, the solution is $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

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4. (10pts) Find the determinant of

$$A=\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 0 & 3 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & -1 & 7 & 2 \end{bmatrix}.$$

[Solution]
By using Laplace expansion, we have

$$\begin{aligned}
\det(A) &= \det\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 0 & 3 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & -1 & 7 & 2 \end{bmatrix} \\
&= -2\det\begin{bmatrix} 1 & 2 & -5 \\ -2 & 3 & 1 \\ 0 & 7 & 2 \end{bmatrix} + (-1)\det\begin{bmatrix} 1 & 2 & -5 \\ -2 & 3 & 1 \\ 3 & 0 & 1 \end{bmatrix} \\
&= (-2)\left(1\det\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} - (-2)\det\begin{bmatrix} 2 & -5 \\ 7 & 2 \end{bmatrix}\right) \\
&\quad + (-1)\left(-2\det\begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} + 3\det\begin{bmatrix} 1 & -5 \\ 3 & 1 \end{bmatrix}\right) \\
&= (-2)\,1\,(6-7) - (-2)(-2)(4+35) \\
&\quad + (-1)(-2)(-2-3) + (-1)\,3\,(1+15) \\
&= -212.
\end{aligned}$$

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5. (10pts) Find the trigonometric function of the form

$$f(t) = c_0 + c_1\sin(t) + c_2\cos(t)$$

that best fits the data points $(0,-1)$, $\left(\frac{\pi}{2},2\right)$, $(\pi,2)$ and $\left(\frac{3\pi}{2},1\right)$, using lease squares.

[Solution]
We want to find a $f(t) = c_0 + c_1\sin(t) + c_2\cos(t)$ such that $f(0) = -1$, $f\left(\frac{\pi}{2}\right) = 2$, $f(\pi) = 2$ and $f\left(\frac{3\pi}{2}\right) = 1$. These conditions give the

system of linear equations

$$\begin{cases} c_0 + c_1 \sin(0) + c_2 \cos(0) = f(0) = -1 \\ c_0 + c_1 \sin\left(\frac{\pi}{2}\right) + c_2 \cos\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 2 \\ c_0 + c_1 \sin(\pi) + c_2 \cos(\pi) = f(\pi) = 2 \\ c_0 + c_1 \sin\left(\frac{3\pi}{2}\right) + c_2 \cos\left(\frac{3\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = 1 \end{cases},$$

or,

$$\begin{cases} c_0 \quad\quad + c_2 = -1 \\ c_0 + c_1 \quad\quad = 2 \\ c_0 \quad\quad - c_2 = 2 \\ c_0 - c_1 \quad\quad = 1 \end{cases}.$$

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$. We can write the system as $A\vec{x} = \vec{b}$. Since $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we have $\ker(A) = \{0\}$. So, the unique least-squares solution of $A\vec{x} = \vec{b}$ is

$$\begin{aligned} \vec{x}^* &= \left(A^T A\right)^{-1} A^T \vec{b} \\ &= \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}. \end{aligned}$$

Moreover, the trigonometric function,

$$f^*(t) = 1 + \frac{1}{2}\sin(t) - \frac{3}{2}\cos(t),$$

best fits the data points $(0, -1)$, $\left(\frac{\pi}{2}, 2\right)$, $(\pi, 2)$ and $\left(\frac{3\pi}{2}, 1\right)$ in the least-squares sense.

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6. (20pts) Find the QR factorization of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{bmatrix}.$$

[Solution]

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Set $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ and $Q = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix}$. By QR decomposition process, we calculate the following:

$$\begin{aligned}
r_{11} &= \|\vec{v}_1\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}, \\
\vec{w}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \\
r_{12} &= \vec{w}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot (-1) + \left(-\frac{1}{\sqrt{2}}\right) \cdot 1 = -\frac{1}{\sqrt{2}}, \\
\vec{u}_2 &= \vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\,\vec{w}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}, \\
r_{22} &= \|\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\,\vec{w}_1\| = \|\vec{u}_2\| = \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{6}}{2}, \\
\vec{w}_2 &= \frac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\,\vec{w}_1}{\|\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\,\vec{w}_1\|} = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{2}{\sqrt{6}} \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \\
r_{13} &= \vec{w}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \cdot 1 + 0 \cdot (-2) + \left(-\frac{1}{\sqrt{2}}\right) \cdot 0 = \frac{1}{\sqrt{2}}, \\
r_{23} &= \vec{w}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{6}} \cdot 1 + \left(-\frac{2}{\sqrt{6}}\right) \cdot (-2) + \left(\frac{1}{\sqrt{6}}\right) \cdot 0 = \frac{5}{\sqrt{6}}, \\
\vec{u}_3 &= \vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3)\,\vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3)\,\vec{w}_2 = \vec{v}_3 - r_{13}\vec{w}_1 - r_{23}\vec{w}_2 \\
&= \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \frac{5}{\sqrt{6}} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned} r_{33} &= \|\vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3)\,\vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3)\,\vec{w}_2\| = \|\vec{u}_3\| \\ &= \sqrt{\left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = \frac{1}{\sqrt{3}}, \\ \vec{w}_3 &= \frac{\vec{v}_3 - r_{13}\vec{w}_1 - r_{23}\vec{w}_2}{\|\vec{v}_3 - r_{13}\vec{w}_1 - r_{23}\vec{w}_2\|} = \frac{\vec{u}_3}{\|\vec{u}_3\|} = \sqrt{3}\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}. \end{aligned}$$

By definition, we have

$$Q = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

and

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{5}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

And, $A = QR$.
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7. (10pts) Find the matrix of the linear transformation,

$$T(M) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} M$$

from $M_{2\times 2}$ to $M_{2\times 2}$ with respect to the basis,

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

and determine whether T is an isomorphism.

[Solution]

Let $M_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ and $M_4 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ form our basis $\mathcal{B}$. And, we have

$$\begin{aligned}
T(M_1) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \\
T(M_2) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
T(M_3) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}, \\
T(M_4) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.
\end{aligned}$$

To get the matrix of T, we have to write them in term of linear combonations of our basis $\mathcal{B}$ first. (Note: Since M_1, M_2, M_3 and M_4 form a basis of our space, the linear combination expression is unique. Therefore if you can find a linear combination expression, then that is the one, no matter how you find it. You can guess, observe, or use system of equations with Gauss-Jordan elimination. Either way is fine if you can find a linear combination expression.) So,

$$\begin{aligned}
[T(M_1)]_{\mathcal{B}} &= (-1)M_1 + 0M_2 + 0M_3 + 0M_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}, \\
[T(M_2)]_{\mathcal{B}} &= 0M_1 + 0M_2 + 0M_3 + 0M_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}, \\
[T(M_3)]_{\mathcal{B}} &= 1M_1 + 0M_2 + 3M_3 + (-2)M_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}_{\mathcal{B}}, \\
[T(M_4)]_{\mathcal{B}} &= 0M_1 + 0M_2 + 0M_3 + (-1)M_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}_{\mathcal{B}}.
\end{aligned}$$

Write them as columns of the matrix B. We have

$$B = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix}.$$

Since, obviously, $\operatorname{rref} B = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I_4$, we know B is not invertible. That means T is not invertible. So, T is not an isomorphism. (Note: you have to use the representation matrix of T which is B to use any theorem we have already derived. Even though, in this case, $T(M) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} M$ is already written as a matrix multiply a variable. But, by definition, $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ is not our B matrix, the representation matrix of T. So, to determine whether T is an isomorphism, you have to use B, not $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$.)

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8. (10pts) Let V and W be linear spaces. Let T be an isomorphism from V to W. Assume that $f_1, f_2, \cdots, f_n$ form a basis of V. Show that $T(f_1), T(f_2), \cdots, T(f_n)$ is a basis of W.

[Solution]

To show that $T(f_1), T(f_2), \cdots, T(f_n)$ are linearly independent, we assume $c_1 T(f_1) + c_2 T(f_2) + \cdots + c_n T(f_n) = 0$. Hence, we have $T(c_1 f_1 + \cdots + c_n f_n) = 0$ since T is a linear transformation. Since T is an isomorphism from V to W, we have T is invertible, that means, $\ker(T) = \{0\}$. This implies $c_1 f_1 + \cdots + c_n f_n = 0$. Moreover, $f_1, f_2, \cdots, f_n$ form a basis of V. This condition forces that $c_1 = c_2 = \cdots = c_n = 0$. So, $T(f_1), T(f_2), \cdots, T(f_n)$ are linearly independent.

Since T is an isomorphism from V to W, we have T^{-1} is an isomorphism from W to V such that $T(T^{-1}(g)) = g$ for all $g \in W$. For all $g \in W$, we have $T^{-1}(g) \in V$. Since $f_1, f_2, \cdots, f_n$ form a basis of V, there exist $t_1, t_2, \cdots, t_n$ such that $T^{-1}(g) = t_1 f_1 + \cdots + t_n f_n$. Therefore, $g = T(T^{-1}(g)) = T(t_1 f_1 + \cdots + t_n f_n) = t_1 T(f_1) + \cdots t_n T(f_n)$. That means g is a linear combination of $T(f_1), T(f_2), \cdots, T(f_n)$. And, obviously, $T(f_1), T(f_2), \cdots, T(f_n)$ are all in W. Thus, we can conclude that $W = \operatorname{span}\{T(f_1), T(f_2), \cdots, T(f_n)\}$.

So, $T(f_1), T(f_2), \cdots, T(f_n)$ form a basis of W

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