110.201 Quiz 3 Solutions: Friday

March 28, 2005

Problem 1

1. Yes, the map is linear. It is not an isomorphism; its kernel contains, for example, \( f(x) = x \), which means the map is not one-to-one

2. The map is linear; this much is obvious. It is also clearly injective, and since an injective linear map \( T : U \rightarrow V \) between linear spaces of the same finite dimension is an isomorphism, the map is an isomorphism.

3. The map is not linear. \( T(kA) = k^2 T(A) \) for all real \( k \).

Problem 2

1. There are several ways to solve this problem. One is the following:

   Let \( B_2 = \{ x^4, 2x^3 - 1, 1 - x^2, 3x - 1, 2x \} \equiv \{ f_1, f_2, f_3, f_4, f_5 \} \). Observe that
   \[
   \begin{align*}
   1 &= (3/2)f_5 - f_4 \quad (1) \\
   x &= f_5/2 \quad (2) \\
   x^2 &= (3/2)f_5 - f_4 - f_3 \quad (3) \\
   x^3 &= f_2/2 \quad (4) \\
   x^4 &= f_1 
   \end{align*}
   \]

   Therefore \( \text{span}\{ f_1, f_2, \ldots, f_5 \} = \text{span}\{ 1, x, x^2, x^3, x^4 \} = P_4 \). It follows immediately that the \( f_i \) are a basis of \( P_4 \), since there are only five of them and \( \dim P_4 = 5 \). The change-of-basis matrix from \( B_1 \) to \( B_2 \) can be found immediately by writing the elements of \( B_2 \) in terms of the standard basis:
\[
\begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
  f_4 \\
  f_5
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 2 & 0 \\
  1 & 0 & -1 & 0 & 0 \\
 -1 & 3 & 0 & 0 & 0 \\
  0 & 2 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
x^2 \\
x^3 \\
x^4
\end{bmatrix}
\]

\( S \) is equal to the \( 5 \times 5 \) matrix on the right side of the above equation.

2. There was a typo on this part: \( T \) should have been written as \( T(p(x)) = p'' + p' + p \). This didn’t happen, so we wound up giving full credit to everyone on this part, and bonus points to anyone who noticed the mistake on their own. Here is a solution to the problem, properly stated. Immediately note that we can write \( T = I + Q \), where \( I \) is the identity transformation and \( Q(p) = p'' + p' \). We know what the matrix for \( I \) looks like, so we’ll just work with \( Q \). Let’s compute the action of \( Q \) on the \( f_i \), using the change of basis matrix above:

\[
\begin{align*}
Q(f_1) &= 2f_2 - 12(f_3 + f_4) + 18f_5 \quad (6) \\
Q(f_2) &= -6(f_3 + f_4) + 21f_5 \quad (7) \\
Q(f_3) &= 2f_4 - 3f_5 \quad (8) \\
Q(f_4) &= -3f_4 + 9f_5/2 \quad (9) \\
Q(f_5) &= -2f_4 + 3f_5 \quad (10)
\end{align*}
\]

Therefore the matrix for \( T \) is \( I_5 \) plus the matrix of \( Q \) in this basis, namely

\[
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 \\
  2 & 1 & 0 & 0 & 0 \\
 -12 & -6 & 1 & 0 & 0 \\
 -12 & -6 & 2 & -2 & -2 \\
 18 & 21 & -3 & 9/2 & 4
\end{bmatrix}
\]

**Problem 3** Writing out the equation defining \( S \) by performing the indicated matrix multiplication:

\[
\begin{bmatrix}
x \\
y \\
0 \\
z
\end{bmatrix} \in S \iff 2x - 3y + 4z = 0
\]
So, letting \( x = s, y = t, z = \frac{2s - 3t}{4} \), it is clear that \( \dim S = 2 \), and that a basis for \( S \) is given by

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -3/4 \end{bmatrix}
\]

To find the dimension of \( S \), you could also have just observed that the equation defining \( S \) is the equation of a plane in \( \mathbb{R}^3 \).

There are a couple of ways to do the second part. The first is to just mess around until you pull an answer out of a hat. The second is a bit sneakier and involves the observation made before: There is an isomorphism of \( S \) onto the plane \( P \) in \( \mathbb{R}^3 \) defined by \( 2x - 3y + 4z = 0 \), namely

\[
\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\]

Therefore, to find a basis of \( S \) such that \( A \) has the given representation in that basis, we can just find a basis of \( P \) such that \( \vec{u} = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \) has the representation. Then just use the above isomorphism to translate that basis back into matrix form. To get a basis like that, pick two vectors \( \vec{v}_1, \vec{v}_2 \) in \( \mathbb{R}^3 \) that are (1) not normal to \( P \) and (2) \( \vec{u} = 2\vec{v}_1 + 3\vec{v}_2 \). Take the projections of \( \vec{v}_1 \) and \( \vec{v}_2 \) onto \( P \), and you get what you need. Take \( \vec{v}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 0 & 0 & -1/3 \end{bmatrix} \). Then their projections onto \( P \) and the corresponding basis matrices are

\[
\text{proj}_P \vec{v}_1 = \frac{1}{15} \begin{bmatrix} 13 \\ 3 \\ -4 \end{bmatrix} \rightarrow \frac{1}{15} \begin{bmatrix} 13 & 3 \\ 0 & -4 \end{bmatrix}
\]

\[
\text{proj}_P \vec{v}_2 = \frac{1}{45} \begin{bmatrix} 2 \\ -3 \\ -9 \end{bmatrix} \rightarrow \frac{1}{45} \begin{bmatrix} 4 & -6 \\ 0 & -7 \end{bmatrix}
\]

One quickly checks that the two matrices above have the required properties.