Solutions Calc III final, May 12, 2005

Problem 1.

$$J_\Phi = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & t \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & r \end{vmatrix} = r^2$$

$$\text{Vol}(C) = \int\int\int (\text{Jacobian})\, dr d\theta dt \int_0^1 \int_0^{2\pi} \int_1^{1/r} r^2 dt d\theta dr = 2\pi \int_0^1 (r - r^2) dr = \frac{\pi}{3}$$

Problem 2.

a) Cavalieri principle: $\int_0^1$ area (slice at z) $dz = \int_0^1 \pi z^2 dz = \frac{\pi}{3}$.

b) Volume between graphs: $\int\int_D \{1 - \sqrt{x^2+y^2}\} dxdy$, where $D = \{x^2 + y^2 \leq 1\}$.

Integral equals (polar coordinates): $\int_0^{2\pi}\int_0^1 (1-r) r dr = \frac{\pi}{3}$.

Problem 3.

Gauss: $\int\int_\Sigma \overrightarrow{r} \cdot dS + \int\int_D \overrightarrow{r} \cdot dS = \int\int_{\partial C} \overrightarrow{r} \cdot dS = \int\int\int_C (\text{div}\, \overrightarrow{r}), dv = 3\text{vol}(C) = \pi$.

Aside: $\int\int_D \overrightarrow{r} \cdot dS = \int\int_D \overrightarrow{r} \cdot \overrightarrow{k}\, dS = \int\int_D dS = \text{area}(D) = \pi$.

Conclusion: $\int\int_\Sigma \overrightarrow{r} \cdot dS = 0$.

Problem 4.

Σ is parametrized by $x = r\cos\theta,\ y = r\sin\theta,\ z = r,\ \theta \in [0, 2\pi],\ r \in [0,1]$.

$T_\theta = (-r\sin\theta, r\cos\theta, 0), \quad T_r = (\cos\theta, \sin\theta, 1)$ and hence

$$T_\theta \times T_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = r(\cos\theta, \sin\theta, -1)$$

Clearly $\overrightarrow{n}_\Sigma = \frac{T_\theta \times T_r}{\|T_\theta \times T_r\|} = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, -1)$ pointing outwards.

$$\begin{aligned} \overrightarrow{F} \cdot dS &= \overrightarrow{F} \cdot (T_\theta \times T_r) d\theta dr \\ &= \left(\frac{r}{r^2 + r^2\cos^2\theta}, 0, \frac{-r\cos\theta}{r^2 + r^2\cos^2\theta}\right) \cdot (r\cos\theta, r\sin\theta, -r) d\theta dr \\ &= \frac{2\cos\theta}{1+\cos^2\theta} d\theta dr \end{aligned}$$

Hence $\int\int_\Sigma \overrightarrow{F} \cdot dS = \int_0^1\int_0^{2\pi} \frac{2\cos\theta}{1+\cos^2\theta} d\theta dr = \int_0^{2\pi} \frac{2\cos\theta}{1+\cos^2\theta} d\theta = \int_0^{2\pi} \frac{2\cos\theta}{2-\sin^2\theta} d\theta$.

With $\sin\theta = u$, $\int \frac{2\cos\theta}{2-\sin^2\theta} d\theta = \int \frac{2du}{2-u^2} = \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+u}{\sqrt{2}-u}\right| = \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right|$.

This yields $\int\int_\Sigma \overrightarrow{F} \cdot dS = \int_0^{2\pi} \frac{2\cos\theta}{2-\sin^2\theta} d\theta = 0$.

Problem 5. Let γ the loop in question, then $\partial\Sigma = -\gamma$.

Stokes: $\int_\gamma \overrightarrow{G} = -\int\int_\Sigma \text{curl}\, \overrightarrow{G} \cdot dS$.

Curl computation (messy): curl $\overrightarrow{G} = \left(\frac{z}{x^2+z^2}, 0, \frac{-x}{x^2+z^2}\right) = \overrightarrow{F}$.

Conclusion: $\int_\gamma \overrightarrow{G} = -\int\int_\Sigma \overrightarrow{F} \cdot dS = 0$.

Problem 6. In spherical coordinates this is determined by $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{3}$.

Area $= \int\int dS = \int_0^{2\pi}\int_0^{\frac{\pi}{3}} \sin\phi\, d\phi\, d\theta = 2\pi \int_0^{\frac{\pi}{3}} \sin\phi\, d\phi = \pi$.

Problem 7. False, since div $\overrightarrow{r} = 3 \neq 0$.

Problem 8.

$$\overrightarrow{H} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ \frac{\partial}{\partial x} g & \frac{\partial}{\partial y} g & \frac{\partial}{\partial z} g \end{vmatrix} = \begin{vmatrix} i & j & k \\ g & g & g \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = -\operatorname{curl}(g, g, g) = -\operatorname{curl}(g\overrightarrow{\omega})$$

Hence $\overrightarrow{J} = -g\overrightarrow{\omega}$.

Problem 9.

Stokes: $\int\int_S \overrightarrow{H} \cdot dS = \int\int_S \operatorname{curl} \overrightarrow{J} \cdot dS = \int_{\partial S} \overrightarrow{J}$.

Whatever parametrizaton $\gamma(t)$ one has for ∂S, the work of $\overrightarrow{J}$ along ∂S is

$$\int \overrightarrow{J}(\gamma(t)) \cdot \gamma'(t)\, dt$$

But $\gamma'(t)$ is in the plane $\{x + y + z = 1\}$ and $\overrightarrow{J} \perp$ plane $\{x + y + z = 1\}$, so the work is zero.