Past Exam Problems in Integrals

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Course 110.202

December 6, 2004

The following is a list of the problems concerning integrals that appeared in the midterm and final exams of Calc III (110.202) within the last several years. You may use them to check your understanding of the relevant material. Some other exam problems may be found at

`http://reserves.library.jhu.edu/access/reserves/findit/exams/110/110202.php`

Note: These problems do not imply, in any sense, my taste or preference for our own exam. Some of the problems here may be more (or less) challenging than what will appear in our exam.

1. Show that there is no vector field $\mathbf{G}$ such that

$$\operatorname{curl}\mathbf{G} = 2x\mathbf{i} + 3yz\mathbf{j} - xz^2\mathbf{k}.$$

   (Hint: Recall that $\operatorname{curl}\mathbf{G}$ is the same as $\nabla \times \mathbf{G}$.)

2. (a) State Green's Theorem.

   (b) Use Green's Theorem to evaluate the contour integral

$$\int_C (1 + y^8)\,\mathrm{d}x + (x^2 + e^y)\,\mathrm{d}y,$$

   where C denotes the boundary of the region enclosed by the curve $y = \sqrt{x}$ and the lines $x = 1$ and $y = 0$.

3. (a) State the Divergence Theorem. Explain briefly what each symbol in the theorem stands for. (You may assume all the differentiability you want.)

(b) Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the boundary surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y = 5$.

4. Let $\mathbf{F}(x, y, z) = (2x + y^2)\mathbf{i} + (2xy + 3y^2)\mathbf{j}$.

   (a) Show that $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

   (b) Find a function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

   (c) Use (b) to evaluate the integral $\int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{s}$, where C is the arc of the curve $y = \sin^3 x$ from $(0, 0)$ to $(\pi/2, 1)$ in the xy-plane.

5. Use Gauss's Divergence Theorem to calculate the total flux out of the cube Ω, given by $-1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1$, of the vector field

$$\mathbf{v}(\mathbf{r}) = 2xy\mathbf{i} + (y - y^2)\mathbf{j} + (x^2 y + z)\mathbf{k}.$$

6. Consider the vector field $\mathbf{G}(\mathbf{r}) = y^2 z\mathbf{i} + (x^2 y + z^2 - 3z)\mathbf{j} + (2yz + e^z)\mathbf{k}$.

   (a) Use Stokes' theorem to express the line integral $\int_C \mathbf{G} \cdot \, \mathrm{d}\mathbf{s}$ as a surface integral, where C denotes the piecewise linear (square) contour that goes from the origin to $(0; 2; 0)$, then to $(0; 2; 2)$, then to $(0; 0; 2)$, and back to the origin.

   (b) Hence evaluate the line integral. [HINT: Do not evaluate the line integral directly unless you have lots of time and want to check your answer.]

7. Consider the iterated integral

$$\int_{-2}^{2} \left(\int_{y^2}^{4} \sqrt{x} y^2 e^{x^3} \, \mathrm{d}x \right) \, \mathrm{d}y.$$

   (a) Sketch the region of integration.

   (b) Reverse the order of integration, by expressing I as an iterated integral with $y$integrated first.

   (c) Hence evaluate I. [WARNING: Do not attempt to evaluate I directly!]