Past Exam Problems in Integrals, Solutions

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Note: These problems do not imply, in any sense, my taste or preference for our own exam. Some of the problems here may be more (or less) challenging than what will appear in our exam.

1. According to the hint (use the hint, please!), if there exists such a $\mathbf{G}$, then we should have

$$\mathrm{div}(\mathrm{curl}\mathbf{G}) = \nabla \cdot (\nabla \times \mathbf{G}) = 0.$$

However, direct computations give us that

$$\mathrm{div}(\mathrm{curl}\mathbf{G}) = \frac{\partial(2x)}{\partial x} + \frac{\partial(3yz)}{\partial y} - \frac{\partial(xz^2)}{\partial z} = 2 + 3z - 2xz \neq 0.$$

This leads to a contradiction. Hence there is no such vector field $\mathbf{G}$.

Remark. You may observe that $\mathrm{div}(\mathrm{curl}\mathbf{G}) = 0$ for some special values of x, y and z. For example, we have $\mathrm{div}(\mathrm{curl}\mathbf{G})(0, 1, -\frac{2}{3}) = 0$. However, what we need is that $\mathrm{div}(\mathrm{curl}\mathbf{G}) = 0$ as a function, i.e. for *every* possible values of x, y and z, and this is where we find the contradiction.

2. (a) **Green's Theorem.** Let D be a simple region and let C be its boundary. Suppose $P : D \to \mathbb{R}$ and $Q : D \to \mathbb{R}$ are of class C^1. Then

$$\int_{C^+} P \,\mathrm{d}x + Q \,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \,\mathrm{d}y.$$

(b) By Green's Theorem, we have

$$\begin{aligned}\int_C (1+y^8)\,\mathrm{d}x + (x^2+e^y)\,\mathrm{d}y &= \iint_D (2x-8y^7)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^{\sqrt{x}} (2x-8y^7)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 (2x^{\frac{3}{2}} - x^4)\,\mathrm{d}x = \frac{3}{5}.\end{aligned}$$

Remark. Yes, I copied Green's Theorem from our textbook, Page 522. In practice we do not need to remember every word in this theorem; what is really important is the formula itself and the relation between C and D, especially the compatibility of their orientations.

3. (a) **Gauss' Divergence Theorem.** Let W be a symmetric elementary region in space. Denote by ∂W the oriented closed surface that bounds W. Let $\mathbf{F}$ be a smooth vector field defined on W. Then

$$\iiint_W (\nabla\cdot\mathbf{F})\,\mathrm{d}V = \iint_{\partial W} \mathbf{F}\cdot\,\mathrm{d}\mathbf{S},$$

i.e.

$$\iiint_W (\operatorname{div}\mathbf{F})\,\mathrm{d}V = \iint_{\partial W} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S.$$

(b) We have $\nabla\cdot\mathbf{F} = 3y$, so Gauss' Divergence Theorem gives

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_W 3y\,\mathrm{d}V &= \int_{-1}^1 \int_0^{1-x^2} \int_0^5 3y\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \\ &= \int_{-1}^1 (1-x^2)\,\mathrm{d}x \cdot \int_0^5 3y\,\mathrm{d}y = \frac{4}{3}\times\frac{75}{2} = 50.\end{aligned}$$

Remark. Again I copied Gauss' Divergence Theorem from our textbook, this time from Page 564, and again what is really important is the formula itself and the relation between W and ∂W. Another important but subtle thing is the requirement for $\mathbf{F}$. In some cases (like here) it has to be smooth everywhere, but in some cases (like in the three-dimensional Stokes' Theorem) it may have finitely many exceptional points. A practical attitude on this issue is that we always keep ourselves on the safe side and play with only smooth vector fields; if it has any exceptional points on the surface or in the solid region, then check the precise statement of the corresponding theorem before our computations. Do not blindly apply these theorems in such cases.

4. (a) Direct computation shows that curl$\mathbf{F} = \mathbf{0}$.

   (b) For the potential function, we may start from $(0,0,0)$ to get

$$
\begin{aligned}
f(x,y,z) &= \int_0^x \mathbf{F}(t,0,0)\cdot(1,0,0)\ \mathrm{d}t + \int_0^y \mathbf{F}(x,t,0)\cdot(0,1,0)\ \mathrm{d}t \\
&\quad + \int_0^z \mathbf{F}(x,y,t)\cdot(0,0,1)\ \mathrm{d}t \\
&= \int_0^x 2t\ \mathrm{d}t + \int_0^y (2xt+3t^2)\ \mathrm{d}t = x^2 + xy^2 + y^3.
\end{aligned}
$$

   (c) Since $\mathbf{F}$ is conservative, we have

$$
\int_C \mathbf{F}\cdot\ \mathrm{d}\mathbf{s} = \mathbf{F}(\pi/2,1,0) - \mathbf{F}(0,0,0) = \frac{\pi^2}{4} + \frac{\pi}{2} + 1.
$$

5. By Gauss' Divergence Theorem, we have

$$
\begin{aligned}
\iint_S \mathbf{v}\cdot\ \mathrm{d}\mathbf{S} = \iiint_\Omega \operatorname{div}\mathbf{v}\ \mathrm{d}V \\
&= \int_{-1}^1\int_{-1}^1\int_{-1}^1 (2y+1-2y+1)\ \mathrm{d}V \\
&= 2\int_{-1}^1\int_{-1}^1\int_{-1}^1\ \mathrm{d}V = 16.
\end{aligned}
$$

   Remark. Although this problem specifically asks us to use Gauss' Divergence Theorem, it will be a good exercise just directly compute the total flux, which involves six surface integral. Each integral is not hard, but it is not trivial to always choose the right orientation and to stay away from simple mistakes.

6. (a) By Stokes' Theorem, we have

$$
\int_C \mathbf{G}\cdot\ \mathrm{d}\mathbf{s} = \iint_S \mathrm{curl}\mathbf{G}\cdot\ \mathrm{d}\mathbf{S},
$$

where S is the square with vertices at the origin and $(0, 2, 0)$, $(0, 2, 2)$, $(0, 0, 2)$ with orientation pointing to the positive direction of the x-axis.

(b) We have

$$\text{curl}\mathbf{G} = (3, y^2, 2xy - 2yz).$$

To parametrize S, we note that it is spanned by the vectors $\mathbf{v}_1 = (0, 2, 0) - (0, 0, 0) = (0, 2, 0)$ and $\mathbf{v}_2 = (0, 0, 2) - (0, 0, 0) = (0, 0, 2)$, so we may naturally choose the parametrization

$$\Phi(u, v) = u\mathbf{v}_1 + v\mathbf{v}_2 + (0, 0, 0) = (0, 2u, 2v) \qquad 0 \leq u, v \leq 1,$$

i.e. with

$$x(u, v) = 0, \qquad y(u, v) = 2u, \qquad z(u, v) = 2v.$$

Since

$$\Phi_u(u, v) \times \Phi_v(u, v) = (0, 2, 0) \times (0, 0, 2) = (4, 0, 0),$$

this is an orientation-preserving parametrization, so

$$\begin{aligned} \int_C \mathbf{G} \cdot \, \mathrm{d}\mathbf{s} &= \iint_S \text{curl}\mathbf{G} \cdot \, \mathrm{d}\mathbf{S} \\ &= \int_0^1 \int_0^1 (3, 4u^2, -4uv) \cdot (\Phi_u(u, v) \times \Phi_v(u, v)) \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_0^1 \int_0^1 (3, 4u^2, -4uv) \cdot (4, 0, 0) \, \mathrm{d}u \, \mathrm{d}v \\ &= 12 \int_0^1 \int_0^1 \, \mathrm{d}u \, \mathrm{d}v = 12. \end{aligned}$$

Remark. For Part (a), when we describe S, do not forget to mention its orientation. For Part (b), do not forget to check the orientation of our parametrization, and if you draw a diagram beforehand (which I should have done but failed to do), then you may find a simpler parametrization $\Phi(u, v) = (0, u, v)$. Furthermore, is it that difficult to do the line integral directly? Maybe tedious, but not that hard. Wanna give it a try?

7. (a) For this part, I give up. I never know how to draw a graph in LaTeX(the language that this and the other lectures are written with), so please excuse me... Described in words, it is the region bounded by $x = y^2$ (i.e. $y = \pm\sqrt{x}$) and $x = 4$.

   (b) For this part, I can do it. By observing the diagram that is not shown here, we have

$$\int_{-2}^{2} \left(\int_{y^2}^{4} \sqrt{x} y^2 e^{x^3} \, \mathrm{d}x \right) \, \mathrm{d}y = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \sqrt{x} y^2 e^{x^3} \, \mathrm{d}y \, \mathrm{d}x.$$

   (c) We have

$$\begin{aligned} \int_{-2}^{2} \left(\int_{y^2}^{4} \sqrt{x} y^2 e^{x^3} \, \mathrm{d}x \right) \, \mathrm{d}y &= \int_0^4 \sqrt{x} e^{x^3} \left(\int_{-\sqrt{x}}^{\sqrt{x}} y^2 \, \mathrm{d}y \right) \, \mathrm{d}x \\ &= \int_0^4 \sqrt{x} e^{x^3} \left(\left. \frac{y^3}{3} \right|_{-\sqrt{x}}^{\sqrt{x}} \right) \, \mathrm{d}x \\ &= \frac{2}{3} \int_0^4 x^2 e^{x^3} \, \mathrm{d}x \\ &= \frac{2}{9} \left(e^{64} - 1 \right). \end{aligned}$$