Midterm Exam I, Solutions

Calculus III, 110.202

11:00 am – 11:50 am
October 12, 2004

1. (a) We have

$$\rho = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3,$$

$$\sin\theta = \frac{-2}{\sqrt{(-2)^2 + (-2)^2}} = -\frac{\sqrt{2}}{2},$$

$$\cos\theta = \frac{-2}{\sqrt{(-2)^2 + (-2)^2}} = -\frac{\sqrt{2}}{2},$$

$$\cos\phi = \frac{1}{\sqrt{(-2)^2 + (-2)^2 + 1^2}} = \frac{1}{3},$$

so we have

$$\theta = \pi + \arcsin\frac{\sqrt{2}}{2} = \frac{5\pi}{4}; \qquad \phi = \arccos\frac{1}{3}.$$

Hence the spherical coordinate for P_1 is

$$\left(3, \frac{5\pi}{4}, \arccos\frac{1}{3}\right).$$

(b) We have $\overrightarrow{P_1P_2} = (3, 4, 0)$ and $\overrightarrow{P_3P_2} = (2, 2, -4)$, so

$$\overrightarrow{P_1P_2} \cdot \overrightarrow{P_3P_2} = 3 \times 2 + 4 \times 2 + 0 \times (-4) = 14,$$

$$\|\overrightarrow{P_1P_2}\| = \sqrt{3^2 + 4^2 + 0^2} = 5$$

$$\|\overrightarrow{P_3P_2}\| = \sqrt{2^2 + 2^2 + (-4)^2} = \sqrt{24} = 2\sqrt{6}.$$

Hence their angle is

$$\arccos\left(\frac{\overrightarrow{P_1P_2}\cdot\overrightarrow{P_3P_2}}{\|\overrightarrow{P_1P_2}\|\ \|\overrightarrow{P_3P_2}\|}\right)=\arccos\left(\frac{7}{5\sqrt{6}}\right).$$

2. (a) (Implicit Differentiation) We may write the relation as

$$F(x,y,z)=x^3+3y^2+8xz^2-3z^3y-1=0,$$

so implicit differentiation gives

$$\frac{\partial z}{\partial x}=-\frac{\partial F/\partial x}{\partial F/\partial z}=-\frac{3x^2+8z^2}{16xz-9z^2y}.$$

(Chain Rule) We may take the partial derivatives for both sides and apply the chain rule to get

$$3x^2+8z^2+16xz\frac{\partial z}{\partial x}-9z^2y\frac{\partial z}{\partial x}=0,$$

so we have

$$\frac{\partial z}{\partial x}=\frac{3x^2+8z^2}{9z^2y-16xz}.$$

(b) We have

$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\&=3xy\left(\ln(2u+3v)+\frac{2u}{2u+3v}\right)+\frac{3u}{2u+3v}\frac{e^y}{2x+y}\end{aligned}$$

3. (a) We have

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0.$$

In our case,

$$\begin{aligned}f_x(x,y,z)&=2xz+ye^{z-1}, & f_x(3,2,1)&=6+2=8,\\f_y(x,y,z)&=-2y+xe^{z-1}, & f_y(3,2,1)&=-4+3=-1,\\f_z(x,y,z)&=x^2+xye^{z-1}, & f_z(3,2,1)&=9+6=15.\end{aligned}$$

Hence the equation for the tangent plane at $(3,2,1)$ is

$$8(x-3)-(y-2)+15(z-1)=0,$$

i.e. $8x-y+15z=37$.

(b) We have

$$(2,3,5)\times(1,1,2)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & 5\\ 1 & 1 & 2\end{vmatrix}=(1,1,-1),$$

so the unit vector along this direction is

$$\mathbf{u}=\frac{(1,1,-1)}{\sqrt{1^2+1^2+(-1)^2}}=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right).$$

On the other hand, from above we have $\nabla f(3,2,1)=(8,-1,15)$. Hence

$$D_{\mathbf{u}}f(3,2,1)=(8,-1,15)\cdot\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)=-\frac{8}{\sqrt{3}}.$$

4. (a) To begin with, we have

$$\nabla f(x,y)=(10y-6xy-5y^2,10x-3x^2-10xy),$$

so $\nabla f(0,0)=\mathbf{0}$ and $(0,0)$ is a critical point. Now the Hessian matrix is

$$H(f)=\begin{pmatrix}-6y & 10-6x-10y\\ 10-6x-10y & -10x\end{pmatrix},$$

$$D=\det(H(f))(0,0)=-100<0$$

so $(0,0)$ is neither local maximum nor local maximum.

(b) Since

$$\nabla f(x,y)=(2x-3y+5,-3x-2+12y),$$

we have $\nabla f(0,0)\neq\mathbf{0}$ and so $(0,0)$ is an ordinary point, neither a local maximum nor a local minimum.

5. (a) We have

$$\nabla f(x,y)=(2x-2xy,4y-x^2),$$

so the fastest decreasing direction is along the direction of

$$-\nabla f(1,3)=-(-4,11)=(4,-11),$$

so the direction vector is

$$\frac{(4,-11)}{\sqrt{(-4)^2+11^2}}=\left(\frac{4}{\sqrt{137}},-\frac{11}{\sqrt{137}}\right).$$

(b) To locate the critical points, we set

$$\nabla f(x,y) = (2x - 2xy, 4y - x^2) = (0,0),$$

so

$$2x - 2xy = 2x(1-y) = 0, \qquad 4y - x^2 = 0.$$

From the first equation, we have either $x = 0$ or $y = 1$. Now we substitute this into the second equation, then we get $y = 1$ or $x^2 = 4$ correspondingly. Hence the critical points are $(0,0), (2,1), (-2,1)$. Furthermore, we have

$$D(x,y) = \det(H(f)(x,y)) = \begin{vmatrix} 2-2y & -2x \\ -2x & 4 \end{vmatrix} = 8(1-y) - 4x^2.$$

Since $D(0,0) = 8 > 0$ and $f_{xx}(0,0) = 2 > 0$, $(0,0)$ is a local maximum; since $D(2,1) = -16 < 0$, $(2,1)$ is a saddle point; since $D(-2,1) = -16 < 0$, $(-2,1)$ is a saddle point.