Midterm Exam I, Solutions

Calculus III, 110.202

11:00 am – 11:50 am
October 12, 2004

1. (a) We have

\[ \rho = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3, \]
\[ \sin \theta = \frac{-2}{\sqrt{(-2)^2 + (-2)^2}} = -\frac{\sqrt{2}}{2}, \]
\[ \cos \theta = \frac{-2}{\sqrt{(-2)^2 + (-2)^2}} = -\frac{\sqrt{2}}{2}, \]
\[ \cos \phi = \frac{1}{\sqrt{(-2)^2 + (-2)^2 + 1^2}} = \frac{1}{3}, \]

so we have

\[ \theta = \pi + \arcsin \frac{\sqrt{2}}{2} = \frac{5\pi}{4}; \quad \phi = \arccos \frac{1}{3}. \]

Hence the spherical coordinate for \( P_1 \) is

\[ \left( 3, \frac{5\pi}{4}, \arccos \frac{1}{3} \right). \]

(b) We have \( \overrightarrow{P_1P_2} = (3, 4, 0) \) and \( \overrightarrow{P_3P_2} = (2, 2, -4) \), so

\[ \overrightarrow{P_1P_2} \cdot \overrightarrow{P_3P_2} = 3 \times 2 + 4 \times 2 + 0 \times (-4) = 14, \]
\[ \| \overrightarrow{P_1P_2} \| = \sqrt{3^2 + 4^2 + 0^2} = 5 \]
\[ \| \overrightarrow{P_3P_2} \| = \sqrt{2^2 + 2^2 + (-4)^2} = \sqrt{24} = 2\sqrt{6}. \]
Hence their angle is
\[ \arccos \left( \frac{\overrightarrow{P_1P_2} \cdot \overrightarrow{P_3P_2}}{\| \overrightarrow{P_1P_2} \| \| \overrightarrow{P_3P_2} \|} \right) = \arccos \left( \frac{7}{5\sqrt{6}} \right). \]

2. (a) (Implicit Differentiation) We may write the relation as
\[ F(x, y, z) = x^3 + 3y^2 + 8xz^2 - 3z^3y - 1 = 0, \]
so implicit differentiation gives
\[ \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = \frac{3x^2 + 8z^2}{16xz + 9z^2y}. \]
(Chain Rule) We may take the partial derivatives for both sides and apply the chain rule to get
\[ 3x^2 + 8z^2 + 16xz \frac{\partial z}{\partial x} - 9z^2y \frac{\partial z}{\partial x} = 0, \]
so we have
\[ \frac{\partial z}{\partial x} = \frac{3x^2 + 8z^2}{9z^2y - 16xz}. \]

(b) We have
\[ \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \]
\[ = 3xy \left( \ln(2u + 3v) + \frac{2u}{2u + 3v} \right) + \frac{3u}{2u + 3v} e^y 2x + y. \]

3. (a) We have
\[ f_x(x_0, y_0, z_0)(x-x_0) + f_y(x_0, y_0, z_0)(y-y_0) + f_z(x_0, y_0, z_0)(z-z_0) = 0. \]
In our case,
\[ f_x(x, y, z) = 2xz + ye^{-z-1}, \quad f_x(3, 2, 1) = 6 + 2 = 8, \]
\[ f_y(x, y, z) = -2y + xe^{-z-1}, \quad f_y(3, 2, 1) = -4 + 3 = -1, \]
\[ f_z(x, y, z) = x^2 + xy e z^{-1}, \quad f_z(3, 2, 1) = 9 + 6 = 15. \]
Hence the equation for the tangent plane at (3, 2, 1) is
\[ 8(x - 3) - (y - 2) + 15(z - 1) = 0, \]
i.e. \[ 8x - y + 15z = 37. \]
(b) We have
\[ (2, 3, 5) \times (1, 1, 2) = \begin{vmatrix} i & j & k \\ 2 & 3 & 5 \\ 1 & 1 & 2 \end{vmatrix} = (1, 1, -1), \]
so the unit vector along this direction is
\[ \mathbf{u} = \frac{(1, 1, -1)}{\sqrt{1^2 + 1^2 + (-1)^2}} = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right). \]
On the other hand, from above we have \( \nabla f(3, 2, 1) = (8, -1, 15) \). Hence
\[ D_{\mathbf{u}} f(3, 2, 1) = (8, -1, 15) \cdot \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) = -\frac{8}{\sqrt{3}}. \]

4. (a) To begin with, we have
\[ \nabla f(x, y) = (10y - 6xy - 5y^2, 10x - 3x^2 - 10xy), \]
so \( \nabla f(0, 0) = \mathbf{0} \) and \((0, 0)\) is a critical point. Now the Hessian matrix is
\[ H(f) = \begin{pmatrix} -6y & 10 - 6x - 10y \\ 10 - 6x - 10y & -10x \end{pmatrix}, \]
so \( D = \det(H(f))(0, 0) = -100 < 0 \) and \((0, 0)\) is neither local maximum nor local minimum.

(b) Since
\[ \nabla f(x, y) = (2x - 3y + 5, -3x - 2 + 12y), \]
we have \( \nabla f(0, 0) \neq \mathbf{0} \) and \((0, 0)\) is an ordinary point, neither a local maximum nor a local minimum.

5. (a) We have
\[ \nabla f(x, y) = (2x - 2xy, 4y - x^2), \]
so the fastest decreasing direction is along the direction of
\[ -\nabla f(1, 3) = -(-4, 11) = (4, -11), \]
so the direction vector is
\[ \frac{(4, -11)}{\sqrt{(-4)^2 + 11^2}} = \left( \frac{4}{\sqrt{137}}, -\frac{11}{\sqrt{137}} \right). \]
(b) To locate the critical points, we set

\[ \nabla f(x, y) = (2x - 2xy, 4y - x^2) = (0, 0), \]

so

\[ 2x - 2xy = 2x(1 - y) = 0, \quad 4y - x^2 = 0. \]

From the first equation, we have either \( x = 0 \) or \( y = 1 \). Now we substitute this into the second equation, then we get \( y = 1 \) or \( x^2 = 4 \) correspondingly. Hence the critical points are \((0, 0), (2, 1), (-2, 1)\). Furthermore, we have

\[ D(x, y) = \det(H(f)(x, y)) = \begin{vmatrix} 2 - 2y & -2x \\ -2x & 4 \end{vmatrix} = 8(1 - y) - 4x^2. \]

Since \( D(0, 0) = 8 > 0 \) and \( f_{xx}(0, 0) = 2 > 0 \), \((0, 0)\) is a local maximum; since \( D(2, 1) = -16 < 0 \), \((2, 1)\) is a saddle point; since \( D(-2, 1) = -16 < 0 \), \((-2, 1)\) is a saddle point.