Past Exam Problems in Vector Analysis and Partial Derivatives

Prof. Qiao Zhang
Course 110.202

October 6, 2004

The following is a list of the problems concerning vector analysis and partial derivatives that appeared in the midterm and final exams of Calc III (110.202) within the last several years. You may use them to check your understanding of the relevant material.

Note: These problems do not imply, in any sense, my taste or preference for our own exam. Some of the problems here may be more challenging than what will appear in our exam.

1. (a) Decompose the vector $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ as a sum $\mathbf{c} + \mathbf{d}$ of two vectors, where $\mathbf{c}$ is parallel to the vector $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{d}$ is orthogonal to $\mathbf{b}$.

   (b) Find the area of the triangle that has the vectors $\mathbf{a}$ and $\mathbf{b}$ as two of its sides.

2. A particle traces out the curve C. Its position at time t is given by $\mathbf{r}(t) = t\mathbf{i} + (3/2)t^2\mathbf{j} + t^2\mathbf{k}$.

   (a) Find its velocity at any time t.

   (b) Find its speed at any time t.

   (c) Find the coordinates of every point where C intersects the plane $2x + 2y - z = 0$.

3. The line l is given parametrically by the equation $\mathbf{r}(t) = 2\mathbf{i} - 2\mathbf{j} + t(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. The point P_1 has coordinates $(7, 2, 3)$. Find an equation for the plane that contains the line l and the point P_1.

4. A spring S is parametrized by the equation $\mathbf{r}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} + 2t\mathbf{k}$. Let P_2 be the point of S where $t = \pi$.

   (a) Find the unit tangent vector to S at P_2.

   (b) Find a parametric equation for the tangent line to S at P_2.

5. Find the first partial derivatives of the following functions

$$(a)\ f(x,y) = \ln(x + \ln(y)), \qquad (b)\ z = g(x,y);$$

   where the function g is defined implicitly by the equation $x^2 + y^3 + 3xz + z^3 = 0$.

6. Use differentials to approximate the value of $\sqrt{99}\sqrt[3]{124}$, given that $\sqrt{100} = 10$ and $\sqrt[3]{125} = 5$.

7. Suppose z is the function $h(x,y)$ of x and y defined by the equation $z = h(x,y) = 2x^2 + y^2 - 10$.

   (a) Find the direction in which the rate of change of z at the point $(2,2)$ is maximal. What is this maximal rate of change?

   (b) Find an equation of the tangent plane to the surface $z = h(x,y)$ at the point $(2,2,2)$.

   (c) Find an equation of the normal line to the surface $z = h(x,y)$ at $(2,2,2)$.

8. Find the extreme values of the function $f(x,y) = x^2 - 4y^2$ on the region bounded by the ellipse $x^2 + 2y^2 = 1$, and the points where these values are attained.

9. Suppose $u = g(x,y)$, where the differentiable function g is unknown. Put $x = r\cos\theta$ and $y = r\sin\theta$, so that $u = g(r\cos\theta, r\sin\theta) = f(r,\theta)$.

   (a) Express the first-order partial derivatives of f in terms of g.

   (b) Express the second-order partial derivative $f_{\theta\theta}$ in terms of g. (Remember that g is a function of x and y, not r and θ.)

10. The water temperature T at the point (x,y,z) is given by $T(x,y,z) = x^2 + x - xy - yz + z^2$. A fish has reached the point $P = (1,1,1)$.

(a) Find the gradient vector of T at the point P.

(b) What rate of increase of temperature does the fish experience at P if it swims with the velocity vector $5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$?

(c) In which direction from P should the fish swim in order to maximize the rate of increase of temperature? [The answer should be a unit vector.] What is this maximum rate of increase, if the fish has a maximum speed of V?

(d) In which direction from P should the fish swim in order to maximize the rate of decrease of temperature? [The answer should be a unit vector.]