Solutions to Past Exam Problems

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Course 110.202

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Warning: Please use this solution with precaution and at your own risk. Despite my best efforts, there may still be mistakes in this solution; if you detect any, please kindly let me know. You may use this solution as a proof of how careful I am (if there is no mistake) or how important and difficult to be careful (if there are mistakes).

1. (a) Since $\mathbf{c}$ is parallel to $\mathbf{b}$, we may write $\mathbf{c} = t\mathbf{b}$ for some number $t \in \mathbb{R}$. Since $\mathbf{d}$ is orthogonal to $\mathbf{b}$, we have

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{c} + \mathbf{d}) \cdot \mathbf{b} = (t\mathbf{b} + \mathbf{d}) \cdot \mathbf{b} = t\mathbf{b} \cdot \mathbf{b} + \mathbf{d} \cdot \mathbf{b} = t\|\mathbf{b}\|^2.$$

Since

$$\mathbf{a} \cdot \mathbf{b} = 8 + 2 - 3 = 7, \qquad \|\mathbf{b}\|^2 = 2^2 + 1^2 + 3^2 = 14,$$

we have

$$t = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} = \frac{7}{14} = \frac{1}{2}.$$

Hence

$$\mathbf{c} = \frac{1}{2}\mathbf{b} = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}, \qquad \mathbf{d} = \mathbf{a} - \mathbf{c} = 3\mathbf{i} + \frac{3}{2}\mathbf{j} - \frac{5}{2}\mathbf{k},$$

and the decomposition is

$$\mathbf{a} = \left(\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}\right) + \left(3\mathbf{i} + \frac{3}{2}\mathbf{j} - \frac{5}{2}\mathbf{k}\right).$$

(**Note:** Still remember the very first word in the problem? So finding $\mathbf{c}, \mathbf{d}$ alone should be not a complete answer.)

(b) The area of the triangle that has the vectors $\mathbf{a}$ and $\mathbf{b}$ as two of its sides is half of the area of the parallelogram that has $\mathbf{a}, \mathbf{b}$ as two adjacent sides. Hence the area of the triangle is

$$\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\|7\mathbf{i}-14\mathbf{j}\| = \frac{7}{2}\sqrt{5}.$$

2. (a) The velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 3t\mathbf{j} + 2t\mathbf{k}.$$

(b) The speed is

$$\|\mathbf{v}(t)\| = \sqrt{1^2 + (3t)^2 + (2t)^2} = \sqrt{1+13t^2}.$$

(c) Let $\mathbf{r}(t)$ be an intersecting point of C with the plane $2x+2y-z=0$, then we have

$$2x(t) + 2y(t) - z(t) = 2t + 3t^2 - t^2 = 0, \qquad \text{i.e. } 2t(t+1) = 0,$$

so $t = 0$ or $t = -1$. Hence the intersecting points are $\mathbf{r}(0) = (0,0,0)$ and $\mathbf{r}(-1) = -\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}$, and their coordinates are $(0,0,0), (-1, \frac{3}{2}, 1)$ respectively. (**Note:** Here we are asked for the *coordinates* for the two points, so finding these points alone is not sufficient for a complete answer.)

3. Pick up two points on the line l, for example $P_2 = \mathbf{r}(0) = (2,-2,0)$ and $P_3 = \mathbf{r}(1) = (4,0,1)$, then this gives two vectors $\overrightarrow{P_1P_2} = (-5,-4,-3)$ and $\overrightarrow{P_1P_3} = (-3,-2,-2)$ on the plane. Hence the norm vector can be taken as $\mathbf{n} = (-5,-4,-3)\times(-3,-2,-2) = (2,-1,-2)$, and the equation for the plane can be taken as $\mathbf{n}\cdot(\mathbf{r} - \overrightarrow{OP_3}) = 0$, so

$$(2,-1,-2)\cdot(x-4, y, z-1) = 0, \qquad \text{i.e. } 2x - y - 2z = 6.$$

(**Note:** Strictly speaking, we should have checked at the very beginning that P does not lie on the line l. What if P does lie on l?)

4. (a) The velocity vector of S is

$$\mathbf{v}(t) = \mathbf{r}'(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + 2\mathbf{k},$$

so the velocity at P_2 is

$$\mathbf{v}(\pi) = -\mathbf{i} - \pi\mathbf{j} + 2\mathbf{k}.$$

Hence the unit tangent vector to S at P_2 is

$$\frac{\mathbf{v}(\pi)}{\|\mathbf{v}(\pi)\|} = -\frac{1}{\sqrt{5+\pi^2}}\mathbf{i} - \frac{\pi}{\sqrt{5+\pi^2}}\mathbf{j} + \frac{2}{\sqrt{5+\pi^2}}\mathbf{k}.$$

(**Note:** Have you noticed the word "unit"?)

(b) The tangent line to S at P_2 is

$$\begin{aligned}\mathbf{r} &= \mathbf{r}(\pi) + t\mathbf{v}(\pi) = (-\pi\mathbf{i} + 2\pi\mathbf{k}) + t(-\mathbf{i} - \pi\mathbf{j} + 2\mathbf{k}) \\ &= -(t+\pi)\mathbf{i} - \pi t\mathbf{j} + 2(t+\pi)\mathbf{k}.\end{aligned}$$

5. (a) We have

$$\frac{\partial f}{\partial x} = \frac{1}{x+\ln y}, \qquad \frac{\partial f}{\partial y} = \frac{1}{x+\ln y}\frac{1}{y} = \frac{1}{y(x+\ln y)}.$$

(b) Write $F(x,y,z) = x^2 + y^3 + 3xz + z^3$, then implicit differentiation gives

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x+3z}{3x+3z^2}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2}{3x+3z^2} = -\frac{y^2}{x+z^2}.$$

(**Note:** Please do not forget the final simplification of $\partial z/\partial y$!)

6. We may take $f(x,y) = \sqrt{x}\sqrt[3]{y}$, then by the linear approximation around the point $(100, 125)$ we have

$$f(x,y) \approx f(100,125) + f_x(100,125)(x-100) + f_y(100,125)(y-125).$$

Simple computations show that

$$f_x(x,y) = \frac{1}{2\sqrt{x}}\sqrt[3]{y}, \qquad f_y(x,y) = \frac{\sqrt{x}}{3(\sqrt[3]{y})^2}.$$

so

$$f(100,125) = \sqrt{100}\sqrt[3]{125} = 10 \times 5 = 50,$$

$$f_x(100,125) = \frac{1}{4}, \qquad f_y(100,125) = \frac{10}{3\times 25} = \frac{2}{15}.$$

Hence

$$\begin{aligned}\sqrt{99}\sqrt[3]{124} = f(99,124) &\approx f(100,125) - f_x(100,125) - f_y(100,125)\\ &= 50 - \frac{1}{4} - \frac{2}{15} = 50 - \frac{23}{60} \approx 50 - 0.38 = 49.62.\end{aligned}$$

(**Note:** On comparison, numerical computation gives $\sqrt{99}\sqrt[3]{124} = 49.61635151....$)

7. (a) The direction with maximal rate of change of $z = h(x,y)$ at $(2,2)$ is the direction of $\nabla h(2,2)$, and the maximal rate is $\|\nabla h(2,2)\|$. We have $\nabla h(x,y) = (4x, 2y)$, so $\nabla h(2,2) = (8,4)$. Hence the direction is $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, and the rate is $4\sqrt{5}$.

   (b) The tangent plane to the surface $z = h(x,y)$ at the point $(2,2,2)$ is given by

   $$z - 2 = h_x(2,2)(x-2) + h_y(2,2)(y-2) = 8(x-2) + 4(y-2),$$

   i.e. $8x + 4y - z = 22$.

   (c) We recall that a normal vector for the tangent space at $(2,2,2)$ is given by $(h_x(2,2), h_y(2,2), -1) = (8,4,-1)$, so an equation for the normal line is $\mathbf{r}(t) = (2,2,2) + t(8,4,-1)$, i.e.

   $$\begin{cases} x = 2 + 8t \\ y = 2 + 4t \\ z = 2 - t. \end{cases}$$

8. Firstly, we locate the critical points. From $\nabla f(x,y) = \mathbf{0}$, we have

   $$f_x(x,y) = 2x = 0, \qquad f_y(x,y) = -8y = 0,$$

   so this gives a critical point $(0,0)$. Hence the critical points include the origin $(0,0)$ and the boundary points on the ellipse $x^2 + 2y^2 = 1$.

   Secondly, we evaluate the values at $(0,0)$. Obviously, $f(0,0) = 0$.

Thirdly, we evaluate the values at the boundary points. We use the parametrization of the ellipse

$$x = \cos\theta, \qquad y = \frac{1}{\sqrt{2}}\sin\theta,$$

so $f(x, y)$ can be expressed in the form

$$g(\theta) = x^2 - 4y^2 = (\cos\theta)^2 - 2(\sin\theta)^2,$$

and the critical points for $g(\theta)$ are given by

$$g'(\theta) = -2\sin\theta\cos\theta - 4\sin\theta\cos\theta = -6\sin\theta\cos\theta,$$

so within $[0, 2\pi)$ the critical points are $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and their values are given by

$$g(0) = f(1, 0) = 1, \qquad g\left(\frac{\pi}{2}\right) = f\left(0, \frac{1}{\sqrt{2}}\right) = -2,$$

$$g(\pi) = f(-1, 0) = 1, \qquad g\left(\frac{3\pi}{2}\right) = f\left(0, -\frac{1}{\sqrt{2}}\right) = -2.$$

(**Note:** An easier (and so trickier) way of estimating f along the ellipse comes from the observation that, as $x^2 + 2y^2 = 1$, we have

$$f(x, y) = x^2 - 4y^2 = 3x^2 - 2(x^2 + 2y^2) = 3x^2 - 2.$$

As obviously we have $-1 \le x \le 1$, the maximal value of f along the ellipse is when $x = \pm 1$ (and so $y = 0$), with value $f(\pm 1, 0) = 3 \times (-1)^2 - 2 = 1$; the minimum value of f along the ellipse is when $x = 0$ (and so $y = \pm\frac{1}{\sqrt{2}}$), with value $f\left(0, \pm\frac{1}{\sqrt{2}}\right) = 3 \times 0^2 - 2 = -2$.)

Finally, we compare the values of f at all critical points. Hence the maximal value of $f(x, y)$ is 1, and it is attained at $(1, 0)$ and $(-1, 0)$; the minimal value of $f(x, y)$ is -2, and it is attained at $(0, \frac{1}{\sqrt{2}})$ and $(0, -\frac{1}{\sqrt{2}})$.
(**Note:** In the above discussion, we have implicitly used the fact that $f(x, y)$ does have extremal values over this region. This is obvious, as the region is compact, i.e. bounded and including all the boundary points. In this course, we may safely assume that the functions in consideration always have extremal values over given regions. However, in practice we cannot expect it to be true in all cases.)

9. (a) We have

$$\frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta,$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial u}{\partial \theta} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial g}{\partial x}r\sin\theta + \frac{\partial g}{\partial y}r\cos\theta.$$

(b) We have

$$\begin{aligned}\frac{\partial^2 f}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(-\frac{\partial g}{\partial x}r\sin\theta + \frac{\partial g}{\partial y}r\cos\theta\right) \\ &= -\frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial x}r\sin\theta\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial y}r\cos\theta\right).\end{aligned} \tag{*}$$

For the first term in (*), we use the product rule to get

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial x}r\sin\theta\right) &= \frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial x}\right)r\sin\theta + \frac{\partial g}{\partial x}\frac{\partial(r\sin\theta)}{\partial \theta} \\ &= \frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial x}\right)r\sin\theta + \frac{\partial g}{\partial x}r\cos\theta.\end{aligned}$$

Since the chain rule gives

$$\frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)\frac{\partial y}{\partial \theta} = -\frac{\partial^2 g}{\partial x^2}r\sin\theta + \frac{\partial^2 g}{\partial x \partial y}r\cos\theta,$$

we have

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial x}r\sin\theta\right) &= \left(-\frac{\partial^2 g}{\partial x^2}r\sin\theta + \frac{\partial^2 g}{\partial x \partial y}r\cos\theta\right)r\sin\theta + \frac{\partial g}{\partial x}r\cos\theta \\ &= -\frac{\partial^2 g}{\partial x^2}r^2(\sin\theta)^2 + \frac{\partial^2 g}{\partial x \partial y}r^2\sin\theta\cos\theta + \frac{\partial g}{\partial x}r\cos\theta.\end{aligned}$$

Similarly, for the second term in (*), we have

$$\frac{\partial}{\partial \theta}\left(\frac{\partial g}{\partial y}r\cos\theta\right) = -\frac{\partial^2 g}{\partial x \partial y}r^2\sin\theta\cos\theta + \frac{\partial^2 g}{\partial y^2}r^2(\cos\theta)^2 - \frac{\partial g}{\partial y}r\sin\theta.$$

Hence

$$\begin{aligned}\frac{\partial^2 f}{\partial\theta^2} &= \frac{\partial^2 g}{\partial x^2}r^2(\sin\theta)^2 - \frac{\partial^2 g}{\partial x\partial y}r^2\sin\theta\cos\theta - \frac{\partial g}{\partial x}r\cos\theta - \frac{\partial^2 g}{\partial xy}r^2\sin\theta\cos\theta \\ &+ \frac{\partial^2 g}{\partial y^2}r^2(\cos\theta)^2 - \frac{\partial g}{\partial y}r\sin\theta \\ &= \frac{\partial^2 g}{\partial x^2}r^2(\sin\theta)^2 - 2\frac{\partial^2 g}{\partial x\partial y}r^2\sin\theta\cos\theta + \frac{\partial^2 g}{\partial y^2}r^2(\cos\theta)^2 - \frac{\partial g}{\partial x}r\cos\theta - \frac{\partial g}{\partial y}r\sin\theta.\end{aligned}$$

10. (a) We have

$$\nabla T = (2x + 1 - y, -x - z, -y + 2z),$$

so the gradient vector of T at P is $\nabla T(1,1,1) = (2,-2,1)$.

(b) The rate of increase is in fact the directional directive along the direction of $5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The direction vector is

$$\mathbf{u} = \frac{5\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\|5\mathbf{i} + 2\mathbf{j} - \mathbf{k}\|} = \frac{(5,2,-1)}{\sqrt{30}},$$

and the directional derivative is

$$D_{\mathbf{u}}T(1,1,1) = \nabla T(1,1,1)\cdot\mathbf{u} = (2,-2,1)\cdot\frac{(5,2,-1)}{\sqrt{30}} = \frac{10-4-1}{\sqrt{30}} = \frac{5}{\sqrt{30}}.$$

Hence the rate of increase is

$$D_{\mathbf{u}}T(1,1,1)\cdot\|5\mathbf{i} + 2\mathbf{j} - \mathbf{k}\| = \frac{5}{\sqrt{30}} \times \sqrt{30} = 5.$$

(c) The direction of maximal rate of increase should be the direction of the gradient, so the direction vector should be

$$\frac{\nabla T(1,1,1)}{\|\nabla T(1,1,1)\|} = \frac{(2,-2,1)}{\|(2,-2,1)\|} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right).$$

As the fish has a maximum speed of V, the maximum rate of increase is

$$V\|\nabla T(1,1,1)\| = V\|(2,-2,1)\| = 3V.$$

(d) The direction of maximal rate of decrease should be negative of the direction of the gradient, so the direction vector should be

$$-\frac{\nabla T(1,1,1)}{\|\nabla T(1,1,1)\|} = \left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right).$$