Solution to Midterm Exam II

Calculus III, 110.202

November 17, 2004

1. (a) We have

$$\iint_D x^2y \, \mathrm{d}A = \int_0^1 \int_{y-1}^{1-y} x^2y \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 y \frac{(1-y)^3 - (y-1)^3}{3} \, \mathrm{d}y$$
$$= \frac{2}{3} \int_0^1 (y - 3y^2 + 3y^3 - y^4) \, \mathrm{d}y = \frac{1}{30}.$$

(b) The iterated integral can be written as a double integral over the unit quarter disc D in the fourth quadrant

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 xy^2\sqrt{x^2+y^2} \, \mathrm{d}y \, \mathrm{d}x = \iint_D xy^2\sqrt{x^2+y^2} \, \mathrm{d}A$$

so we may use the polar coordinate to get

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 xy^2\sqrt{x^2+y^2} \, \mathrm{d}y \, \mathrm{d}x$$
$$= \int_0^1 \int_{-\frac{\pi}{2}}^0 (r\cos\theta)(r\sin\theta)^2\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} r \, \mathrm{d}\theta \, \mathrm{d}r$$
$$= \int_0^1 r^5 \, \mathrm{d}r \int_{-\frac{\pi}{2}}^0 \cos\theta(\sin\theta)^2 \, \mathrm{d}\theta = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18}.$$

Remark. We may also change the order of integration to compute this integral. However, check the following argument to see where goes wrong.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 xy^2\sqrt{x^2+y^2}\ \mathrm{d}y\ \mathrm{d}x = \int_{-1}^0 \int_0^{\sqrt{1-y^2}} xy^2\sqrt{x^2+y^2}\ \mathrm{d}x\ \mathrm{d}y$$

$$\overset{u=x^2+y^2}{=\!=\!=\!=} \frac{1}{2}\int_{-1}^0 \int_{y^2}^1 y^2\sqrt{u}\ \mathrm{d}u\ \mathrm{d}y = \frac{1}{2}\int_{-1}^0 y^2\left(\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=y^2}^{u=1}\right)\ \mathrm{d}y$$

$$= \frac{1}{3}\int_{-1}^0 y^2\left(1-y^3\right)\ \mathrm{d}y = \frac{1}{3}\left(\frac{y^3}{3}-\frac{y^6}{6}\right)\Big|_{y=-1}^{y=0} = \frac{1}{6}.$$

Think about it!

2. (a) Since the parallelogram is bounded by the lines $y-x=0$, $y-x=2$, $x+y=2$ and $x+y=4$, we may use the change of variables

$$u = y-x, \qquad v = x+y,$$

so that the boundary curves become $u=0$, $u=2$, $v=2$ and $v=4$, and we also have

$$x = \frac{v-u}{2}, \qquad y = \frac{u+v}{2}, \qquad \frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u = -\frac{1}{2}.$$

Hence

$$\iint_D \frac{x-y}{x+y}\ \mathrm{d}A = \int_0^2 \int_2^4 \frac{-u}{v}\frac{1}{2}\ \mathrm{d}u\ \mathrm{d}v = -\frac{1}{2}\int_0^2 u\ \mathrm{d}u \int_2^4 \frac{\mathrm{d}v}{v} = -\ln 2.$$

Another approach is to use the formula

$$\iint_D f(x,y)\ \mathrm{d}x\ \mathrm{d}y$$
$$= \int_0^1 \int_0^1 f((b_1-a_1)u + (c_1-a_1)v + a_1, (b_2-a_2)u + (c_2-a_2)v + a_2)$$
$$\times \left|(b_1-a_1)(c_2-a_2) - (b_2-a_2)(c_1-a_1)\right|\ \mathrm{d}u\ \mathrm{d}v,$$

where D is the parallelogram with vertices (a_1,a_2), (b_1,b_2), (c_1,c_2) and (d_1,d_2). In our case, we can show that the parallelogram has vertices at $(1,1)$, $(0,2)$, $(2,2)$ and $(1,3)$, so our formula gives (by taking[1] $(a_1, a_2) = (1,1)$, $(b_1, b_2) = (0,2)$ and $(c_1, c_2) = (2,2)$)

$$\iint_D \frac{x-y}{x+y}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^1 \int_0^1 \frac{(-u+v+1)-(u+v+1)}{(-u+v+1)+(u+v+1)} |(-1)-1|\,\mathrm{d}u\,\mathrm{d}v$$
$$= \int_0^1 \int_0^1 \frac{-2u}{2v+2} \times 2\,\mathrm{d}u\,\mathrm{d}v = -\ln 2.$$

(b) Set $u = x - y$ and $v = x + y$, then the boundary curves for D can be written as $u^2 + v = 0$ and $v + 4 = 0$. Furthermore, we have

$$x = \frac{u+v}{2}, \qquad y = \frac{v-u}{2}, \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

Hence

$$\iint_D x\,\mathrm{d}x\,\mathrm{d}y = \int_{-2}^{2} \int_{-4}^{-u^2} \frac{u+v}{2}\frac{1}{2}\,\mathrm{d}v\,\mathrm{d}u = \int_{-2}^{2} \left(-2 + u - \frac{u^3}{4} + \frac{u^4}{8}\right) \mathrm{d}u$$
$$= -\frac{32}{5}.$$

3. (a) We have

$$L(\mathbf{c}) = \int_0^2 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,\mathrm{d}t = \int_0^2 \sqrt{1 + 4t^2 + 4t^4}\,\mathrm{d}t$$
$$= \int_0^2 (1 + 2t^2)\,\mathrm{d}t = \left(t + \frac{2}{3}t^3\right)\Big|_0^2 = \frac{22}{3}.$$

(b) We have

$$\int_{\mathbf{c}} \mathbf{F} \cdot\,\mathrm{d}\mathbf{s} = \int_0^2 \Big(x(t), 2y(t), 3z(t)\Big) \cdot \Big(x'(t), y'(t), z'(t)\Big)\,\mathrm{d}t$$
$$= \int_0^2 \Big(t, 2t^2, 2t^3\Big) \cdot \Big(1, 2t, 2t^2\Big)\,\mathrm{d}t$$
$$= \int_0^2 \Big(t + 4t^3 + 4t^5\Big)\,\mathrm{d}t = \frac{182}{3}.$$

[1] Although it does not matter which vertex is (a_1, a_2), once it is determined its opposite vertices has to be (d_1, d_2).

4. (a) Note that the boundary S consists of two parts, i.e. the lower-half unit sphere and the unit disc on the xy-plane.

For the lower-half unit sphere, we have the parametrization

$$\Phi : [0, 2\pi]\times\left[\frac{\pi}{2}, \pi\right] \to \mathbb{R}^3, \qquad (\theta, \phi) \mapsto (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi),$$

so

$$\|\Phi_\theta \times \Phi_\phi\| = \|(-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi)\| = \sin\phi.$$

Hence its surface area is

$$A_1 = \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \|\Phi_\theta \times \Phi_\phi\| \, \mathrm{d}\phi \, \mathrm{d}\theta = \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta = 2\pi.$$

For the unit disc on the xy-plane, we have the spherical coordinate parametrization

$$\Psi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3, \qquad (r, \theta) \mapsto (r\cos\theta, r\sin\theta, 0),$$

so

$$A_2 = \int_0^1\int_0^{2\pi} \|\Psi_r \times \Psi_\theta\| \, \mathrm{d}\theta \, \mathrm{d}r = \int_0^1\int_0^{2\pi} r \, \mathrm{d}\theta \, \mathrm{d}r = \pi.$$

Hence the surface area of S is $A_1 + A_2 = 3\pi$.

(b) Using the spherical coordinates, we have

$$\Phi_\theta \times \Phi_\phi = (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi).$$

The point $(0, 0, -1)$ is given by $\phi = \pi$, i.e. $\Phi(\theta, \pi) = (0, 0, 1)$, so

$$\Phi_\theta(\theta, \pi)\times\Phi_\phi(\theta, \pi) = (-\cos\theta\sin^2\pi, -\sin\theta\sin^2\pi, -\sin\pi\cos\pi) = \mathbf{0}.$$

Hence S is not regular at the point $(0, 0, -1)$.

For the tangent plane, we identify the unit sphere as $F(x, y, z) = 0$, where $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Hence the tangent plane is given by

$$F_x(0, 0, -1)(x-0)+F_y(0, 0, -1)(y-0)+F_z(0, 0, -1)(z-(-1)) = 0.$$

After the simplification, this is reduced to $z = -1$.

5. (a) To compute the volume of W, we regard it as the volume above the graph $f(x,y) = -\sqrt{1-x^2-y^2}$ with $x^2+y^2 \le 1$, so

$$\begin{aligned}
\mathrm{Vol}(W) &= \iiint_W \mathrm{d}V = \iint_{x^2+y^2\le 1} \left(\int_{-\sqrt{1-x^2-y^2}}^{0} \mathrm{d}z \right) \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^1 \int_0^{2\pi} \left(\int_{-\sqrt{1-r^2}}^{0} \mathrm{d}z \right) r\, \mathrm{d}\theta\, \mathrm{d}r = 2\pi \int_0^1 r\sqrt{1-r^2}\, \mathrm{d}r \\
&\overset{u=r^2}{=\!=} \pi \int_0^1 \sqrt{1-u}\, \mathrm{d}u = \frac{2\pi}{3}.
\end{aligned}$$

Remark. A more natural approach is to apply the spherical coordinates, but then one has to quote the Jacobian for the spherical coordinates in the formula.

(b) Although it is not easy to visualize this solid region, at least we know that the projection of this solid region on the xy-plane is the unit disc $x^2+y^2 \le 1$ because this is so for one of the cylinders $x^2+y^2=1$. Furthermore, for each point (x,y) in this disc, then the solid region goes all the way up (resp. down) until it touches the other cylinder $y^2+z^2=1$, so we have the range $-\sqrt{1-y^2} \le z \le \sqrt{1-y^2}$. Hence

$$\begin{aligned}
\mathrm{Vol}(W) &= \iiint_W \mathrm{d}V = \iint_{x^2+y^2\le 1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \mathrm{d}z \right) \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^{2\pi} \int_0^1 \left(\int_{-\sqrt{1-r^2\sin^2\theta}}^{\sqrt{1-r^2\sin^2\theta}} \mathrm{d}z \right) r\, \mathrm{d}r\, \mathrm{d}\theta \\
&= 2\int_0^{2\pi} \int_0^1 r\sqrt{1-r^2\sin^2\theta}\, \mathrm{d}r\, \mathrm{d}\theta \\
&\overset{u=1-r^2\sin^2\theta}{=\!=\!=} \int_0^{2\pi} \int_{\cos^2\theta}^{1} \frac{\sqrt{u}}{\sin^2\theta}\, \mathrm{d}u\, \mathrm{d}\theta \\
&= \frac{2}{3}\int_0^{2\pi} \frac{1-(\cos^2\theta)^{\frac{3}{2}}}{\sin^2\theta}\, \mathrm{d}\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \frac{1-\cos^3\theta}{\sin^2\theta}\, \mathrm{d}\theta = \frac{16}{3}.
\end{aligned}$$

Remark. In the last step, we use the symmetry to convert an integral over $[0, 2\pi]$ to one over $[0, \frac{\pi}{2}]$ to guarantee that $\cos\theta$ is positive, so that $(\cos^2\theta)^{\frac{3}{2}} = \cos^3\theta$ (what if $\cos\theta < 0$?)