Solution to Past Exam Problems in Integrals

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Note: These problems do not imply, in any sense, my taste or preference for our own exam. Some of the problems here may be more (or less) challenging than what will appear in our exam.

Important Note: These solutions also serve as examples for "professional" mathematical writings. I am already lazy enough, so in exams do not try to present a shorter solution for such problems.

1. If we regard the triangle as a y-simple region, then

$$\begin{aligned}\iint_D x^2y^2 \,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_0^{2x} x^2y^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x^2 \left(\frac{y^3}{3}\Big|_{y=0}^{y=2x} \right) \mathrm{d}x \\ &= \frac{8}{3}\int_0^1 x^5 \,\mathrm{d}x = \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}.\end{aligned}$$

If we regard the triangle as an x-simple region, then

$$\begin{aligned}\iint_D x^2y^2 \,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \int_{\frac{y}{2}}^1 x^2y^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^2 y^2 \left(\frac{x^3}{3}\Big|_{x=\frac{y}{2}}^{x=1} \right) \mathrm{d}y \\ &= \frac{1}{3}\int_0^1 y^2 \left(1 - \frac{y^3}{8}\right) \mathrm{d}y = \frac{1}{3}\left(\frac{8}{3} - \frac{64}{48}\right) = \frac{4}{9}.\end{aligned}$$

Remark. This double integral is easy, as the vertices are already explicitly given. How about over the triangle enclosed by $y = 2x$, $x = 1$ and $y = 0$?

2. (a) By the cylindrical coordinates, we have

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Hence the shape of the platform is given by

$$r^2\cos^2\theta + r^2\sin^2\theta \le (2-z)^2, \qquad 0 \le z \le 1,$$

i.e.

$$r^2 \le (2-z)^2, \qquad 0 \le z \le 1,$$

i.e.

$$r + z \le 2, \qquad 0 \le z \le 1.$$

(b) Let W be the solid region of this platform, then it is z-simple and we have

$$V = \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \left(\iint_{x^2+y^2\le(2-z)^2} \mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}z.$$

For the inner double integral, we apply the polar coordinates, then the integration region is, by Part (a), given by the formula $r^2 \le (2-z)^2$, i.e. $r \le 2-z$. Note that here we have used the fact that $0 \le z \le 1$ (so that $2-z > 0$). Hence

$$V = \int_0^1 \left(\int_0^{2\pi} \int_0^{2-z} r\,\mathrm{d}r\,\mathrm{d}\theta \right) \mathrm{d}z = \int_0^1 \pi(2-z)^2\,\mathrm{d}z = \frac{7\pi}{3}.$$

Remark. A more natural approach to Part (b) is to use the cylindrical coordinates for the triple integral. However, since we did not officially discuss the formula for triple integrals in cylindrical coordinates, here we have to use an indirect approach. Well, as it is said in China, whatever its color might be, a cat is good as long as it catches rats. Also, such combinations of z-simpleness and two-dimensional changes of variables may prove useful in many computations of triple integrals.

3. To begin with, we should draw a diagram from which we can see easily that the integration region D should be the region on the xy-plane bounded by the curve given by $1 + x^2 + y^2 = 2x^2 + 2y^2$, and the height

at (x, y) should be $(1 + x^2 + y^2) - 2(x^2 + y^2)$ (this depends on which surface is above and which is below), so the volume of the solid is

$$V = \iint_D \Big((1 + x^2 + y^2) - 2(x^2 + y^2)\Big)\, \mathrm{d}x\, \mathrm{d}y.$$

Now the condition $1 + x^2 + y^2 = 2x^2 + 2y^2$ gives $x^2 + y^2 = 1$, so D should be the unit disc, and using the polar coordinates we have

$$V = \int_0^{2\pi} \int_0^1 \Big((1 + r^2) - 2r^2)\Big) r\, \mathrm{d}r\, \mathrm{d}\theta = 2\pi \int_0^1 (1 - r^2) r\, \mathrm{d}r = \frac{\pi}{2}.$$

Remark. Here pay attention to the determination of the integration region. As always, seeing is believing!

4. Since $z = 1 - x - y$, the plane itself is parametrized by

$$(x, y) \mapsto (x, y, 1 - x - y) : \mathbb{R}^2 \to \mathbb{R}^3.$$

The intersection of the plane with the cylinder lies above the disc

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

which can be parametrized by

$$(r, \theta) \in [0, 1] \times [0, 2\pi] \mapsto (r \cos\theta, r \sin\theta).$$

Thus

$$\Phi : (r, \theta) \mapsto (r \cos\theta, r \sin\theta, 1 - r(\cos\theta + \sin\theta))$$

does the trick.

The area of the surface is given by $\iint |\Phi_r \times \Phi_\theta|\, \mathrm{d}r\, \mathrm{d}\theta$, with the limits of integration given above. I won't write out the calculation of the cross-product here; the result if $r(1, 1, 1)$, so the area is

$$\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (\sqrt{3}) r\, \mathrm{d}r\, \mathrm{d}\theta = \frac{\sqrt{3}}{2} \cdot 2\pi \cdot \left(r^2|_{r=0}^{r-1}\right) = (\sqrt{3})\pi.$$

Remark. This solution is copied word-by-word from the official solution of that exam (including its typos), and you may have tasted different flavors of different instructors even in writing solutions. Here the point is to determine what the intersection should look like (is it on the plane or on the cylinder?), and to see this, as always, we have to draw a diagram!

5. By definition, the arc length is

$$L = \int_0^{\frac{\pi}{4}} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ \mathrm{d}t = \int_0^{\frac{\pi}{4}} \sqrt{(-4\sin(2t))^2 + (4\cos(2t))^2 + 3^2}\ \mathrm{d}t$$
$$= \int_0^{\frac{\pi}{4}} \sqrt{16(\sin^2(2t) + \cos^2(2t)) + 9}\ \mathrm{d}t = \int_0^{\frac{\pi}{4}} \sqrt{16+9}\ \mathrm{d}t = \frac{5}{4}\pi.$$

Remark. This computation is very straightforward; if you are waiting for some surprises, sorry to have made you disappointed...

6. As before, the first thing is to draw a diagram from which we easily observe that the volume is given by

$$V = \iint_D \Big(2y - (x^2 + y^2)\Big)\ \mathrm{d}x\ \mathrm{d}y,$$

where D is the region on the xy-plane bounded by the curve $x^2 + y^2 = 2y$. We may transform it to be

$$x^2 + y^2 - 2y = 0, \qquad \text{i.e. } x^2 + (y-1)^2 = 1,$$

so it is a circle with center $(0,1)$ and radius 1. Therefore D is the disc with center $(0,1)$ and radius 1. Now we use the polar coordinate for D

$$x = r\cos\theta, \qquad y = r\sin\theta + 1$$

with $0 \le r \le 1, 0 \le \theta \le 2\pi$ (note that we have shifted y by 1 because here the center is not the origin any more). Hence

$$V = \int_0^{2\pi} \int_0^1 \Big(2(r\sin\theta + 1) - (r\cos\theta)^2 - (r\sin\theta + 1)^2\Big) r\ \mathrm{d}r\ \mathrm{d}\theta.$$

Remark. One may ask (if you do not, then you should) why the Jacobian is still r, even when we use a different polar coordinate change of variables. To see this, compute it! Also, why "DO NOT EVALUATE!"? Evaluate it! The answer is $\frac{\pi}{2}$.

7. We change the order of integrations to get

$$\int_0^1 \int_y^1 \cos\left(\frac{1}{2}\pi x^2\right)\ \mathrm{d}x\ \mathrm{d}y = \int_0^1 \int_0^x \cos\left(\frac{1}{2}\pi x^2\right)\ \mathrm{d}y\ \mathrm{d}x = \int_0^1 x\cos\left(\frac{1}{2}\pi x^2\right)\ \mathrm{d}x$$
$$\overset{u=\frac{\pi}{2}x^2}{=\!=\!=} \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\ \mathrm{d}u = \frac{1}{\pi}.$$

Remark. As you may guess, the point is the correct determination of the integration limits after the change of order of integration, and for this we need to draw a diagram to visualize the actual shape of the integration region.

8. We have

$$V = \iint_D (x^2 + 2y)\ \mathrm{d}x\ \mathrm{d}y.$$

From the diagram (which you should have drawn the first second you read the problem) we may observe that the integration region D is bounded by $x = 0$ (as we are restricted to the first quadrant), $x+y = 2$ (the line passing through $(2, 0)$ and $(0, 2)$) and $y = 4 - x^2$, so

$$V = \int_0^2 \int_{2-x}^{4-x^2} (x^2 + 2y)\ \mathrm{d}y\ \mathrm{d}x = \int_0^2 (x^3 - 7x^2 + 4x + 12)\ \mathrm{d}x = \frac{52}{3}.$$