02/22/05

Yesterday

Definition: The plane

$$z = f(a,b) + m(x-a) + n(y-b)$$

approximates to first order the graph of f @ (a,b) iff

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - [f(a,b) + m(x-a) + n(y-b)]}{\|(x,y) - (a,b)\|} = 0$$

Observation 1.

If such a "well-approximating" plane exists, then it is unique, and

$$\begin{cases} m = \dfrac{\partial f}{\partial x}(a,b) \\[2ex] n = \dfrac{\partial f}{\partial y}(a,b) \end{cases}$$

Example

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = \begin{cases} \dfrac{x y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Note that f is continuous everywhere.

[Check continuity @ $(0,0)$:

$$0 \le |f(x,y)| \le \left| \frac{xy^2}{y^2} \right| = |x|$$

Pinching lemma $\Rightarrow$ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

$$0 \qquad \lim_{(x,y)\to(0,0)} |x| = 0$$

]

Q Is there a well-approximating plane to the graph of f @ $(0,0)$?

A If such a plane exists, its equation is given by

$$z = f(0,0) + m(x-0) + n(y-0),$$

where

$$\begin{cases} m = \dfrac{\partial f}{\partial x}(0,0) = \displaystyle\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0 \\[2ex] n = \dfrac{\partial f}{\partial y}(0,0) = \displaystyle\lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0 \end{cases}$$

<u>Conclusion</u>: If such a plane exists, it must be the plane

$$z = 0. \qquad \text{(the } xy\text{-plane)}.$$

Verify the condition:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - 0}{\|(x,y)\|} \overset{?}{=} 0,$$

in other words

$$\lim_{(x,y)\to(0,0)} \frac{\frac{xy^2}{x^2+y^2}}{(x^2+y^2)^{1/2}} = \lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}} \overset{?}{=} 0$$

FALSE!

$\cancel{(x,y) \to \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}$

$$x = t$$
$$y = t, \qquad t \to 0$$

$$\lim_{t \to 0} \frac{t \cdot t^2}{(t^2+t^2)^{3/2}} = \frac{1}{2^{3/2}} \neq 0 \quad !$$

Therefore, if a well approximating plane existed, it would have been $z = 0$.

But this one does not do the job.

Conclusion $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(a,b) \in \mathbb{R}^2$ iff the following two conditions are satisfied:

α) $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ exist. ~~Ex~~

β)
$$\lim_{(x,y) \to (a,b)} \frac{f(x,y) - \left\{ f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) \right\}}{\| (x,y) - (a,b) \|} = 0$$

Compare to the situation of $g: \mathbb{R} \to \mathbb{R}$.

If $\displaystyle \frac{dg}{dt}(3) = \lim_{t \to 0} \frac{g(t+3) - g(t)}{t}$ exists (α),

then.
$$\lim_{x \to 3} \frac{g(x) - \left\{ g(3) + \frac{dg}{dt}(3) \cdot (t-3) \right\}}{\cancel{t}\; x-3}$$

$$= \lim_{x \to 3} \left(\frac{g(x) - g(3)}{x-3} - \frac{dg}{dt}(3) \right)$$

$$= \left(\lim_{x \to 3} \frac{g(x) - g(3)}{x-3} \right) - \frac{dg}{dt}(3)$$

$$= \frac{dg}{dt}(3) - \frac{dg}{dt}(3) = 0$$

So condition β) is automatically satisfied!

<u>Explanation</u> $\cancel{f: \mathbb{R}^2 \to \mathbb{R}}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

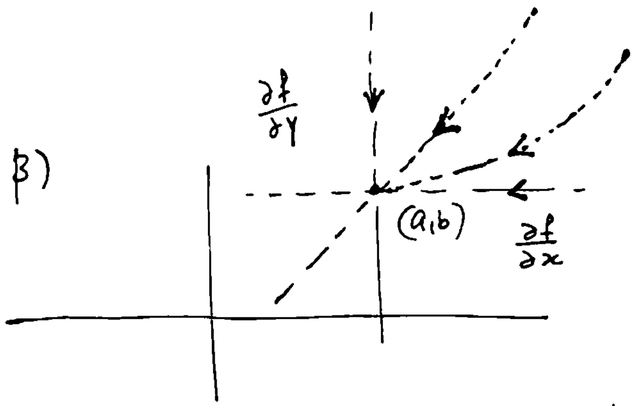
$g : \mathbb{R} \longrightarrow \mathbb{R}$

α) exist !

$$\begin{cases} \dfrac{\partial f}{\partial x}(a,b) = \lim\limits_{t \to 0} \dfrac{f(a+t,b) - f(a,b)}{t} \\ \dfrac{\partial f}{\partial y}(a,b) = \lim\limits_{t \to 0} \dfrac{f(a,b) - f(a,b+t)}{t} \end{cases}$$

α) $g'(a) = \lim\limits_{t \to a} \dfrac{g(t+a) - g(a)}{t}$.

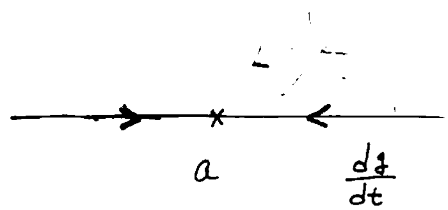
β)



More ways to approach (a,b) !

Several possible directions
[Infinitely many!]

not necessarily "compatible"
with the directions
along which you compute
$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.



Only one direction,
Compatible with
Computing $\dfrac{dg}{dt}(a)$

Similarly for $f : \mathbb{R}^3 \to \mathbb{R}$

<u>Definition</u>

f is differentiable at $\underline{\underline{a}} = (x_0, y_0, z_0) \in \mathbb{R}^3$

<u>iff</u> the following two conditions

are satisfied:

α) $\quad \dfrac{\partial f}{\partial x}(\underline{\underline{a}}),\ \dfrac{\partial f}{\partial y}(\underline{\underline{a}}),\ \dfrac{\partial f}{\partial z}(\underline{\underline{a}})$ exist

β) $$\lim_{(x,y,z)\to \underline{\underline{a}}} \frac{f(x,y,z) - \left\{ f(x_0,y_0,z_0) + \frac{\partial f}{\partial x}(a)(x-x_0) + \frac{\partial f}{\partial y}(\underline{\underline{a}})(y-y_0) + \frac{\partial f}{\partial z}(z-z_0)\right.}{\| (x,y,z) - \underline{\underline{a}} \|}$$

$$= 0.$$

GENERAL CASE : $f : \mathbb{R}^m \to \mathbb{R}^n$.

1. Matrices and linear maps.

Let $A = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1m} \\ \vdots & & \\ a_{n1} & a_{n2} \dots & a_{nm} \end{pmatrix}$ ~~a~~ $n \times m$ matrix.

Then A determines a MAP.

$$[A] : \mathbb{R}^m \to \mathbb{R}^n \quad \underline{\text{OR}} \quad T_A : \mathbb{R}^m \to \mathbb{R}^n$$

given by $T_A(x_1, \dots, x_m) = (y_1, \dots, y_n)$

where $A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

Example 1) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix}$ $\quad 2 \times 3$ matrix

$T_A : \mathbb{R}^3 \to \mathbb{R}^2$

$$T_A(x, y, z) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ 2x + y + z \end{pmatrix} = (x - z, 2x + y + z)$$

2) $A = (3)$. $T_A : \mathbb{R} \to \mathbb{R}$, $T_A(x) = 3x$

i.e. $[(3)] : \mathbb{R} \to \mathbb{R}$, $[(3)](x) = 3x$.

Proposition

$A = n \times m$ matrix, $\quad B = k \times n$ matrix

$$\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n \xrightarrow{T_B} \mathbb{R}^k$$

Then $\quad T_B \circ T_A = T_{BA}$

where $BA =$ matrix multiplication of B and A

Main Feature (check !)

LINEARITY $\begin{cases} T_A(\underline{x} + \underline{y}) = T_A(\underline{x}) + T_A(\underline{y}), & \forall\ \underline{x}, \underline{y} \in \mathbb{R}^m \\ T_A(c\,\underline{x}) = c\, T_A(\underline{x}), & \forall\ c \in \mathbb{R},\ \underline{x} \in \mathbb{R}^m \end{cases}$

Proposition [Homework]

Prove that any map $T : \mathbb{R}^m \to \mathbb{R}^n$ which satisfies the linearity conditions is of the type $\quad T = T_A$, with $A =$ some $n \times m$ matrix.

2. Definition of Differentiability

Definition 1 $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $a \in \mathbb{R}^m$ iff there exists a linear map $T: \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - T(x-a)}{\| x - a \|} = 0 .$$

We set $\boxed{f'(a) = T}$ if such a T exists.

Observation 2 If such a linear map exists, then $T = [Df(a)]$,

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a), \ldots, & \frac{\partial f_1}{\partial x_m}(a) \\ \vdots & \\ \frac{\partial f_n}{\partial x_1}(a), \ldots, & \frac{\partial f_n}{\partial x_m}(a) \end{pmatrix}$$

is the matrix of partial derivatives.

Based on this observation $\longrightarrow$

<u>Definition 2</u> $f: \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $a \in \mathbb{R}^m$ iff :

α) $\dfrac{\partial f_j}{\partial x_i}(a)$ exist, $1 \le i \le m$, $1 \le j \le n$.

β) $$\lim_{x \to a} \frac{f(x) - f(a) - [Df(a)](x-a)}{\| x - a \|} = 0$$

We say that $f'(a)$ is the <u>linear map</u>

$[Df(a)] : \mathbb{R}^m \to \mathbb{R}^n$.

matrix of partial derivatives.

A FUNCTION $f : \mathbb{R}^m \to \mathbb{R}^n$

is DIFF AT ~~$x_0 \in \mathbb{R}^m$~~

$x_0 \in \mathbb{R}^m$ IFF

f can be approximated (to first order)

by a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$

near ~~an~~ x_0.

If ~~f is diff.~~ such a T exists,

it is given by the matrix of partial

derivatives at x_0:

$$T = [Df(x_0)].$$

<u>Example</u>

Let M a $n \times m$ matrix, and

$f = T_M : \mathbb{R}^m \to \mathbb{R}^n$, $f(x) = Mx$, $x \in \mathbb{R}^m$.

Prove that f is differentiable at $a \in \mathbb{R}^m$.

$\longrightarrow$ need to show that there exists

linear map $T_A : \mathbb{R}^m \to \mathbb{R}^n$, A= $n \times m$ matrix

such that

$$\lim_{x \to a} \frac{T_M(x) - T_M(a) - T_A(x-a)}{\| x - a \|} = 0$$

$$\Updownarrow$$

$$\lim_{x \to a} \frac{T_M(x-a) - T_A(x-a)}{\| x - a \|} = 0$$

$\Rightarrow$ SIMPLY TAKE $A = M$!

Therefore $f'(a) = [M] = f$

<u>Answer</u> $\forall$ $a \in \mathbb{R}^m$, $f'(a) = f$, when f is a linear map.

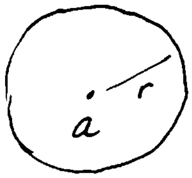
Main Tool:

$f = (f_1, .., f_n)$

<u>Theorem</u> $C^1 \Rightarrow$ DIFF

A) Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $a \in \mathbb{R}^m$.

$B_r(a)$



IF $\dfrac{\partial f_i}{\partial x_j}$ exist, $(1 \le i \le n,\ 1 \le j \le m)$

and are <u>continuous</u> in a ball $B_r(a)$, for some $r > 0$

[i.e. "near" a]

then f is ~~cont~~ <u>differentiable</u> at a.

B) If $\dfrac{\partial f_i}{\partial x_j}$ exist and are

continuous everywhere, $\dfrac{\partial f_i}{\partial x_j} : \mathbb{R}^m \to \mathbb{R}$

then f is differentiable everywhere.

Example

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad f(x,y) = x \sin y$

$$\frac{\partial f}{\partial x}(x,y) = \sin y$$

$$\frac{\partial f}{\partial y}(x,y) = x \cos y.$$

Clearly (WHY ?)

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous everywhere.

$\Rightarrow$ f is continuous everywhere.

II. Main Application of Differentiability:

1) Computing Directional Derivatives.

Let $f: \mathbb{R}^3 \to \mathbb{R}$, $a \in \mathbb{R}^3$.

A ~~[illegible] f is~~

For $0 \neq \vec{v} \in \mathbb{R}^3$,

<u>Definition</u>

$$\partial_{\vec{v}} f(a) := \lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t}$$

$$\frac{\partial f}{\partial \vec{v}}(a)$$

$$D_{\vec{v}} f(a)$$

provided the limit exist.

Geometric Interpretation: Rate of change of f in the $\vec{v}$ direction.

<u>NB</u> preferably $\vec{v}$ is taken to be a unit vector.

Theorem

Assume $f: \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^3$.

Let $0 \neq \vec{v} \in \mathbb{R}^3$ a non-zero vector.

Then $\partial_{\vec{v}} f(a)$ exists, and is given by the formula:

$$\boxed{\partial_{\vec{v}} f(a) = \nabla f(a) \cdot \vec{v}} = \frac{\partial f}{\partial x}(a) \cdot v_1 + \frac{\partial f}{\partial y}(a) v_2 + \frac{\partial f}{\partial z}(a) v_3$$

Proof. f differentiable at a, means

$$\lim_{x \to a} \frac{f(x) - f(a) - [Df(a)] \cdot (x-a)}{\| x-a \|} = 0.$$

Let $x = a + tv$, $t \to 0$, get:

$$\lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a) - [Df(a)] \cdot (t\vec{v})}{\| t\vec{v} \|} = 0$$

$$\iff \lim_{t \to 0} \left(\frac{f(a+t\vec{v}) - f(a)}{t} - [Df(a)](\vec{v}) \right) = 0$$

$$\iff \lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t} = [Df(a)](\vec{v})$$

$$\lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t} = \partial_{\vec{v}} f(a)$$

$$[Df(a)](\vec{v}) = \left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a), \frac{\partial f}{\partial z}(a) \right) \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \nabla f(a) \cdot \vec{v}$$

<u>Note</u> If f differentiable at $a \in \mathbb{R}^3$,

$$\partial_{\vec{v}} f(a) = \frac{\partial f}{\partial x}(a) \cdot v_1 + \frac{\partial f}{\partial y}(a) \cdot v_2 + \frac{\partial f}{\partial z}(a) \cdot v_3$$

DIRECTIONS KNOW ABOUT EACH OTHER!

2) Directions of ~~maximal~~ greatest. increase ..

$\vec{V}_+$ = unit vector such that

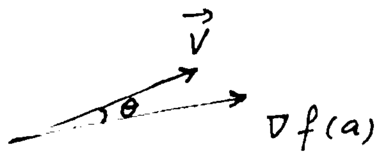
$$\partial_{\vec{V}} f = \text{maximum}$$

$$\|$$

$$\nabla f(a) \cdot \vec{V} = \| \nabla f(a) \| . \| \vec{V} \| . \cos\theta$$

$$= \| \nabla f(a) \| . \cos\theta$$

maximum if $\theta = 0$



i.e. $\boxed{\vec{V}_+ = \dfrac{\nabla f(a)}{\|\nabla f(a)\|}}$ Direction of greatest increase

Similarly, direction of greatest decrease is

$$\vec{V}_- = - \frac{\nabla f(a)}{\|\nabla f(a)\|}$$

Corollary The greatest possible ~~[illegible]~~ rate of change along a unit vector is

$$\partial_{\vec{V}_+} f(a) = \nabla f(a) \cdot \vec{V}_+ = \nabla f(a) \cdot \frac{\nabla f(a)}{\|\nabla f(a)\|} = \frac{\|\nabla f(a)\|^2}{\|\nabla f(a)\|} = \|\nabla f(a)\|$$

Corollary

The greatest possible increase (rate of change) along a unit vector is :

$$\partial_{\vec{V}_+} f(a) = \nabla f(a) \cdot \vec{V}_+$$

$$= \nabla f(a) \cdot \frac{\nabla f(a)}{\|\nabla f(a)\|}$$

$$= \frac{\|\nabla f(a)\|^2}{\|\nabla f(a)\|}$$

$$= \|\nabla f(a)\|$$

$$= \sqrt{\left(\frac{\partial f}{\partial x}(a)\right)^2 + \left(\frac{\partial f}{\partial y}(a)\right)^2 + \left(\frac{\partial f}{\partial z}(a)\right)^2}$$