SUMMARY

1. **Definition** $f : \mathbb{R}^m \to \mathbb{R}^n$, $f = (f_1, \dots, f_n)$ is differentiable at $a \in \mathbb{R}^m$ iff the following two conditions are satisfied:

α) $\dfrac{\partial f_i}{\partial x_j}(a)$ exist, $1 \le i \le n$, $1 \le j \le m$

β) $$\lim_{x \to a} \frac{f(x) - f(a) - [Df(a)](x-a)}{\|x-a\|} = 0,$$

where $[Df(a)] : \mathbb{R}^m \to \mathbb{R}^n$ is the linear map determined by the matrix of partial derivatives computed at a:

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_m}(a) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(a) & \dots & \frac{\partial f_n}{\partial x_m}(a) \end{pmatrix}$$

Notation $f'(a) = [Df(a)]$

so $\begin{cases} f'(a) : \mathbb{R}^m \to \mathbb{R}^n \text{ is a linear map} \\ Df(a) = \text{"just" a matrix } (n \times m) \end{cases}$

Example If $F : \mathbb{R}^m \to \mathbb{R}^n$ is linear, i.e. $F(X) = B \cdot X$, where $B = (n \times m)$-matrix, then

$$F'(a) = F, \quad \forall\, a \in \mathbb{R}^m$$

2. If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $a = (x_0 y_0 z_0) \in \mathbb{R}^3$,
then $Df(a) = \left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a), \frac{\partial f}{\partial z}(a) \right)$, hence
the linear map $f'(a) : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f'(a)(\vec{v}) = \nabla f(a) \cdot \vec{v}$$

3. <u>THEOREM</u> $[C^1 \Rightarrow \text{DIFF}]$ $f = (f_1, \dots, f_n) : \mathbb{R}^m \to \mathbb{R}^n$

If $\frac{\partial f_i}{\partial x_j}$ exist ~~and~~ $(1 \le i \le n,\ 1 \le j \le m)$
<u>AND</u> are continuous at a (everywhere),
<u>THEN</u> f is differentiable at a (everywhere).

4. <u>Example</u> $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x \sin y$.
Then $\frac{\partial f}{\partial x}(x,y) = \sin y$, $\frac{\partial f}{\partial y}(x,y) = x \cos y$, hence
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} : \mathbb{R}^2 \to \mathbb{R}$ are continuous everywhere, so f is differentiable everywhere.

<u>Note</u> $Df(x,y) = (\sin y, x \cos y)$

$f'(x,y) : \mathbb{R}^2 \to \mathbb{R}$, $f'(x,y)(\vec{v}) = \nabla f(x,y) \cdot \vec{v}$.

Example. For $(x,y) = (1,0)$,

$f'(1,0) : \mathbb{R}^2 \to \mathbb{R}$, $f'(1,0)(\vec{v}) = \vec{j} \cdot \vec{v}$

4. Directional Derivatives

a. Definition For $\vec{v} \neq 0$ and $f : \mathbb{R}^m \to \mathbb{R}^n$, $a \in \mathbb{R}^m$,

$$\partial_{\vec{v}} f(a) = \lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t}$$

(rate of change along $\vec{v}$).

b) THEOREM If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at a, then for every $\vec{v} \neq 0$, $\boxed{\partial_{\vec{v}} f(a) = f'(a)(\vec{v})}$

c) Particular case.

Say $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^3$.

Then, for $\vec{v} \neq 0$,

$$\boxed{\partial_{\vec{v}} f(a) = \nabla f(a) \cdot \vec{v}}$$

5. Direction of greatest increase.

Let $\vec{v}$ a unit vector in $\mathbb{R}^3$, $\|\vec{v}\| = 1$, and $f : \mathbb{R}^3 \to \mathbb{R}$ differentiable at $\underline{a} \in \mathbb{R}^3$. We can write

$$\partial_{\vec{v}} f(a) = \nabla f(a) \cdot \vec{v} = \|\nabla f(a)\| \cdot \|\vec{v}\| \cdot \cos\theta$$

$$= \|\nabla f(a)\| \cdot \cos\theta$$

where $\theta = \measuredangle(\nabla f(a), \vec{v})$

Therefore $\partial_{\vec{v}} f(a)$ is maximal when $\theta = 0$, in other words for the direction

$$\vec{V}_{max} = \frac{\nabla f(a)}{\|\nabla f(a)\|}$$

(the unit vector along $\nabla f(a)$).

The actual value of this greatest rate of change is

$$\nabla f(a) \cdot \vec{v}_{max} = \| \nabla f(a) \|$$

$$= \sqrt{\left(\frac{\partial f}{\partial x}(a)\right)^2 + \left(\frac{\partial f}{\partial y}(a)\right)^2 + \left(\frac{\partial f}{\partial y}(a)\right)^2}$$

6. Chain Rule.

Theorem

$$\begin{array}{ccccc} \mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^n & \xrightarrow{g} & \mathbb{R}^k \\ \Downarrow & & \Downarrow & & \Downarrow \\ a & \longmapsto & f(a) & \longmapsto & g(f(a)) \end{array}$$

Assume $f: \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\underline{a} \in \mathbb{R}^m$.

—"— $g: \mathbb{R}^n \to \mathbb{R}^k$ —"— —— $f(a) \in \mathbb{R}^n$.

THEN $f \circ g: \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at a,

and

composition of linear maps

$$\boxed{(f \circ g)'(a) = g'(f(a)) \circ f'(a)}$$

$$\mathbb{R}^m \xrightarrow{f'(a)} \mathbb{R}^n \xrightarrow{g'(f(a))} \mathbb{R}^k$$

In terms of matrices,

matrix multiplication

$$D_{f \circ g}(a) = D_g(f(a)) \cdot D_f(a)$$

Example

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (x^2+y, x+y)$

$g : \mathbb{R}^2 \to \mathbb{R}$

$g(x,y) = xy$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[rr, bend right] & \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$g \circ f : \mathbb{R}^2 \to \mathbb{R}$

a) Compute $D_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 1 \\ 1 & 1 \end{pmatrix}$.

b) Compute $D_g(x,y) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = (y, x)$

Hence $D_g(f(x,y)) = D_g(x^2+y, x+y) = (x+y, x^2+y)$.

d) Compute $D_{g\circ f}(x,y)$.

First, $g\circ f(x,y) = g(f(x,y)) = g(x^2+y, x+y) = (x^2+y)(x+y)$
$= x^3 + x^2y + xy + y^2$

Hence $D_{g\circ f}(x,y) = (3x^2+2xy+y \; , \; x^2+x+2y)$

e) Verify that $D_g(f(x,y)) \cdot D_f(x,y) = D_{g\circ f}(x,y)$:

$$(x+y, x^2+y)\begin{pmatrix} 2x & 1 \\ 1 & 1 \end{pmatrix} = (3x^2+2xy+y, \; x^2+x+2y)$$

7. Important application

Assume $\gamma(t) = (x(t), y(t), z(t)) : \mathbb{R} \to \mathbb{R}^3$ is a differentiable path. (this amounts to $x(t), y(t), z(t)$ differentiable as functions of t).

Let $f : \mathbb{R}^3 \to \mathbb{R}$ a differentiable map.

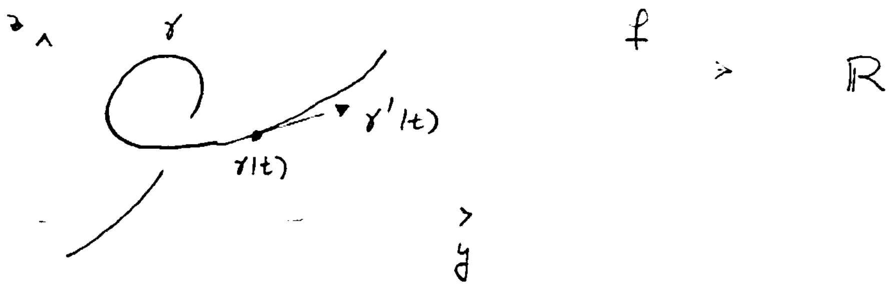
$$\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$
$$\varphi$$

Let $\varphi = f \circ \gamma$, $\varphi(t) = f(\gamma(t))$, then $\varphi : \mathbb{R} \to \mathbb{R}$

Chain Rule implies:

$$\varphi'(t) = D_f(\gamma(t)) \cdot D_\gamma(t)$$

$$\boxed{\varphi'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t)}$$



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Example $\gamma(t) = (\sin t, \cos t, t)$, $f(x,y,z) = xyz$.

For $\varphi(t) = f(\gamma(t)) = \sin t \cdot \cos t \cdot t$, we have:

$\varphi'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t)$. But $\nabla f(x,y,z) = (yz, xz, xy)$,

so $\nabla f(\gamma(t)) = (t \cos t, t \sin t, \sin t \cos t)$

Also $\gamma'(t) = (\cos t, -\sin t, 1)$.

$\varphi'(t) = t \cos^2 t - t \sin^2 t + \sin t \cos t$.

8. Geometric Interpretation

$f : \mathbb{R}^3 \to \mathbb{R}$ differentiable.

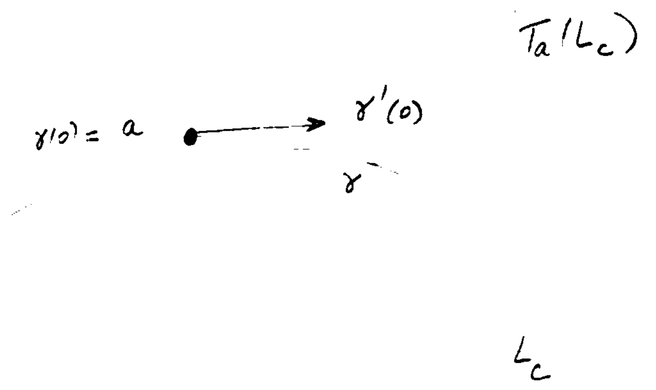
Level sets : $L_c = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = c \}$

Note Given $c \in \mathbb{R}$, L_c = surface in $\mathbb{R}^3$.

Theorem

For $a \in L_c$, $\nabla f(a) \perp \underbrace{T_a(L_c)}_{\text{tangent plane to } L_c \text{ at } a.}$

$\wedge \nabla f(a)$



Note $T_a(L_c)$ = spanned by the vectors $\gamma'(0)$,
where $\gamma(t)$ is a curve inside L_c,
such that $\gamma(0) = a$.

In other words, to prove the theorem we need to prove the following:

Given $\gamma(t) \in L_c$, $\gamma(0) = a$.

Then $\nabla f(a) \perp \gamma'(0)$

Put $\varphi(t) = f(\gamma(t))$.

Since $\gamma(t) \in L_c, \ \forall t \implies \varphi(t) = c, \ \forall t$.

Hence $\varphi'(0) = 0$.

But $\varphi'(0) = \nabla f(\gamma(0)) \cdot \gamma'(0) = \nabla f(a) \cdot \gamma'(0)$ $\Big\} \implies \nabla f(a) \cdot \gamma'(0) = 0$

9. Equation of the tangent plane. $a = (x_0, y_0, z_0)$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ differentiable, $a \in \mathbb{R}^3$, and set $c = f(a)$. Then $a \in L_c$ (surface).

The plane $T_a(L_c)$ passes through a and $\perp \nabla f(a)$.

Equation of $T_a(L_c)$:

$$\frac{\partial f}{\partial x}(a)\,(X - x_0) + \frac{\partial f}{\partial y}(a)\,(Y - y_0) + \frac{\partial f}{\partial z}(a)\,(Z - z_0) = 0$$

10. Tangent plane to the graph of $f : \mathbb{R}^2 \to \mathbb{R}$

The graph of f is the set $\{(x, y, f(x,y)) \mid (x,y) \in \mathbb{R}^2\}$.
It can be seen as the level set L_0 of the function $F : \mathbb{R}^3 \to \mathbb{R}$,

$$F(x,y,z) = f(x,y) - z$$

Therefore the tangent plane @ (x_0, y_0) is

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(X - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(Y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(Z - z_0) = 0$$

$$\iff \frac{\partial f}{\partial x}(x_0, y_0)(X - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(Y - y_0) - (Z - f(x_0, y_0)) = 0$$

$$\iff \boxed{Z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(X - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(Y - y_0)}$$