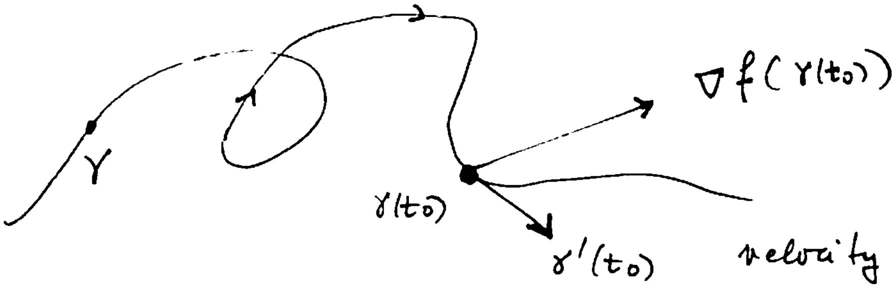
Lecture 12 [Monday, Feb 28]

Level Sets and Tangent planes

① Recall application of chain rule.

$f : \mathbb{R}^2 \to \mathbb{R}$



velocity = tangent to the path.
(why?).

$\varphi(t) = f(\gamma(t))$

dot product

Chain Rule $\Rightarrow$ $\boxed{\varphi'(t) = \nabla f(\gamma(t_0)) \cdot \gamma'(t_0)}$

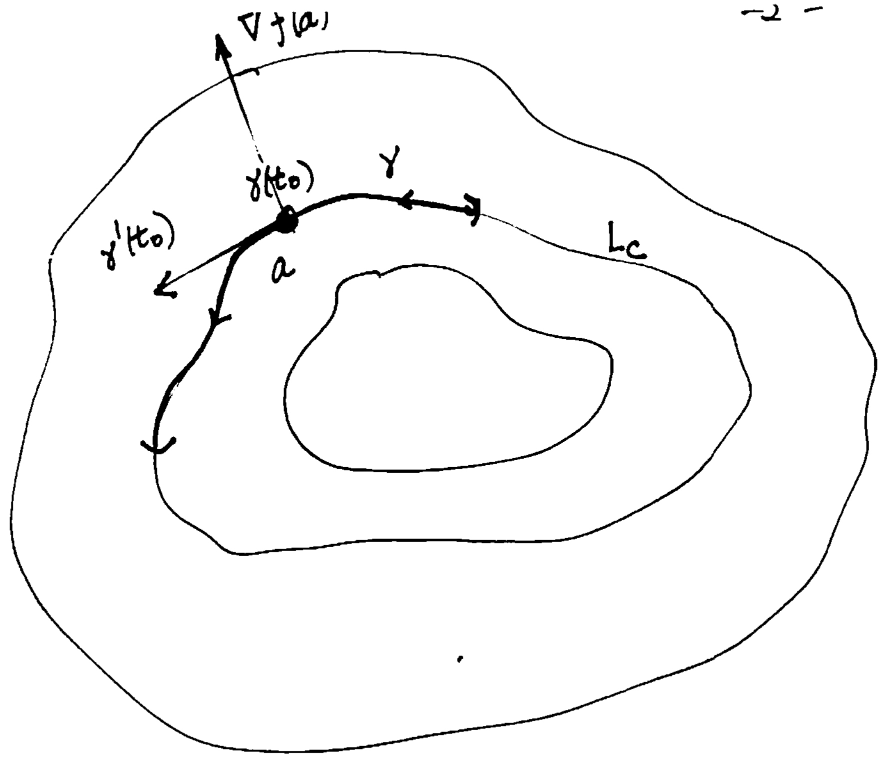
② Level sets.

$f : \mathbb{R}^2 \to \mathbb{R}$. For $c \in \mathbb{R}$,

$L_c = \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) = c \}$

= "curve" in $\mathbb{R}^2$

Q Say $a \in L_c$. Then $C =$?



$f : \mathbb{R}^2 \to \mathbb{R}$

Let $\underset{=}{\overset{(x_0, y_0)}{a}} \in L_c$ and parametrize L_c by $\gamma(t) = (x(t), y(t))$.

(near a), let $a = \gamma(t_0)$,

i.e. $\begin{cases} x_0 = x(t_0) \\ y_0 = y(t_0) \end{cases}$

Let $\varphi(t) = f(\gamma(t))$.

Then $\varphi(t) = c, \ \forall t$

$\Downarrow$

$\varphi'(t) = 0, \ \forall t$, in particular

$\varphi'(t_0) = 0$

$\|$

$\langle \nabla f(\gamma(t_0)), \gamma'(t_0) \rangle = \langle \nabla f(a), \gamma'(t_0) \rangle$

$$\langle \nabla f(a), \gamma'(t_0) \rangle = 0 \iff \boxed{\nabla f(a) \perp \gamma'(t_0)}$$

③ <u>Geometric Interpretation</u>.

$$\nabla f(a) \perp \underbrace{T_a(L_c)}$$

tangent line at $\underline{\underline{a}}$ to curve L_c

<u>Application</u>.

Equation of the line $T_a(L_c)$

$$\begin{cases} a = (x_0, y_0) \in T_a(L_c) \\ \nabla f(a) = \frac{\partial f}{\partial x}(a)\vec{i} + \frac{\partial f}{\partial y}(a)\vec{j} \perp T_a(L_c) \end{cases}$$

Equation:

$$\cancel{\frac{\partial f}{\partial x}(a)(X - x_0) + \frac{\partial f}{\partial y}(a)(Y - y_0) = 0}$$

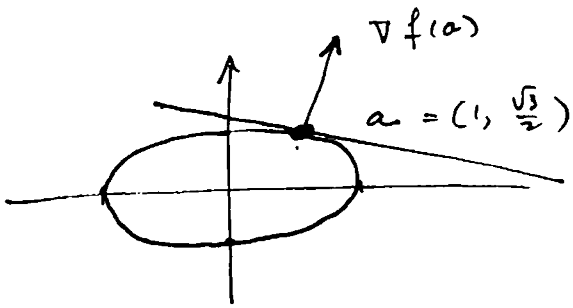
$$\frac{\partial f}{\partial x}(a)(X - x_0) + \frac{\partial f}{\partial y}(a)(Y - y_0) = 0$$

(4) Example.

Consider the curve in $\mathbb{R}^2$ given by

$$\mathcal{E}: \quad \frac{X^2}{4} + Y^2 = 1$$

$$a = \left(1, \frac{\sqrt{3}}{2}\right) \in \mathcal{E}$$



Then $\mathcal{E} = L_1$, $\to$ 1- level set of

$$f(x,y) = \frac{x^2}{4} + y^2 \quad , \quad f: \mathbb{R}^2 \to \mathbb{R}.$$

$a = \left(1, \frac{\sqrt{3}}{2}\right) \in L_1$, so the equation of $T_a(\mathcal{E})$ is:

$$\frac{\partial f}{\partial x}(a)(X-1) + \frac{\partial f}{\partial y}(a)\left(Y - \frac{\sqrt{3}}{2}\right) = 0 \qquad (*)$$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \frac{x}{2} \quad \Rightarrow \quad \frac{\partial f}{\partial x}\left(1, \frac{\sqrt{3}}{2}\right) = \frac{1}{2} \\ \frac{\partial f}{\partial y}(x,y) = 2y \quad \Rightarrow \quad \frac{\partial f}{\partial y}\left(1, \frac{\sqrt{3}}{2}\right) = \sqrt{3}. \end{cases}$$

Eq. $(*)$ becomes:

$$\frac{1}{2}(X-1) + \sqrt{3}\left(Y - \frac{\sqrt{3}}{2}\right) = 0 \quad , \quad \frac{1}{2}(X-1) + \sqrt{3}Y - \frac{3}{2} = 0$$

$$X - 1 + 2\sqrt{3}Y - 3 = 0$$

$$\boxed{X + 2\sqrt{3}Y - 4 = 0}$$