Lecture 13 [Tu, March 1, 2005]

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(I)
- Gradient $\perp$ level set (continuation)
- Equation $T_a(L_c)$
- Example Ellipsoid $\frac{x^2}{4} + y^2 + 2z^2 = 4$.

  $a = (2,1,1) \in \mathcal{E}$, $T_a(\mathcal{E})$

- note $T_a(S) = \{a + \gamma'(t_0) \mid \gamma(t) = \text{path in } S, \gamma(t_0) = a\}$ S = surface

(II) Motion in a (potential) ~~vector~~ field:

$$m\, c''(t) = \nabla V(c(t))$$

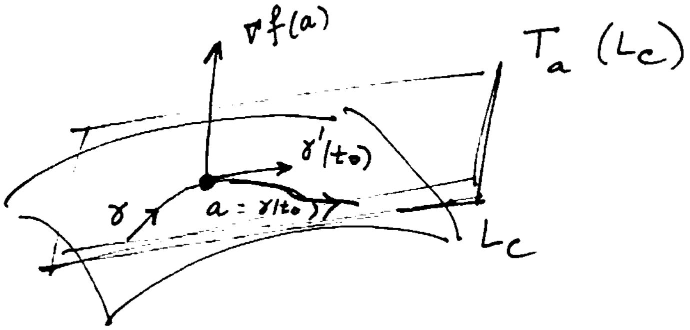
Conservation of energy

$$E = \frac{1}{2} m \|c'(t)\|^2 - V(c(t)). = \text{constant}$$

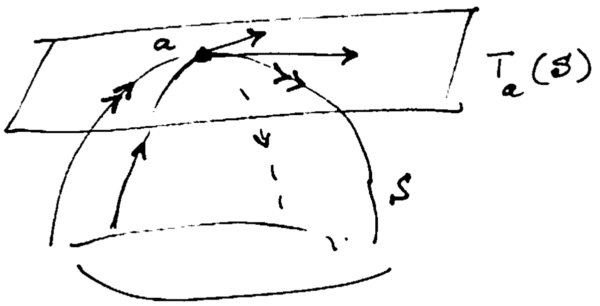
⑤ The case $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

Level sets $L_c = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = c \}$.
(given $c \in \mathbb{R}$)

L_c = "surface" in $\mathbb{R}^3$.



Let $\gamma(t) = (x(t), y(t), z(t))$ a curve in L_c passing ~~thr~~ through $a = (x_0, y_0, z_0)$, say $\begin{cases} x_0 = x(t_0) \\ y_0 = y(t_0) \\ z_0 = z(t_0) \end{cases}$



Let $\varphi(t) = f(\gamma(t))$

Then $\varphi(t) = c, \ \forall t$

$\Downarrow$

$\varphi'(t) = 0, \ \forall t$.

In particular $\varphi'(t_0) = 0$

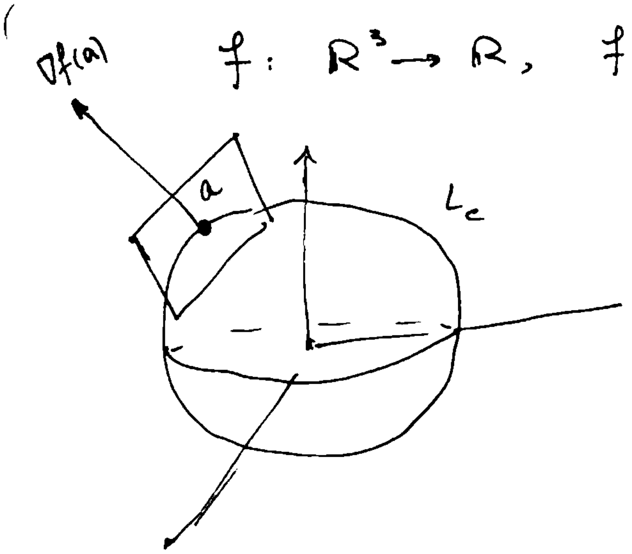
$$\varphi'(t_0) = 0$$
$$\parallel$$
$$\nabla f(a) \cdot \gamma'(t_0) = 0. \quad \Rightarrow \quad \boxed{\nabla f(a) \perp \gamma'(t_0)}$$

Taking different paths through a, $\gamma'(t_0)$ span $T_a(L_c)$.

<u>Conclusion</u>: $\nabla f(a) \perp T_a(L_c)$

L_c = level set passing through a. $[c = \qquad ?]$

Perfect Illustration



$f: R^3 \to R$, $f(\underline{\underline{x}}) = \|\underline{\underline{x}}\|^2 = x^2+y^2+z^2$.

$L_c = \partial B(0, \sqrt{c})$.

For $a \in L_c$,

$\nabla f(a) = 2\underline{\underline{a}}$.

Recall If $a = (x,y,z) \in R^3$, sometimes we use the notation:
$\underline{\underline{a}} = x\vec{i}+y\vec{j}+z\vec{k} = \overrightarrow{Oa}$ to distinguish "vectors" from "points".

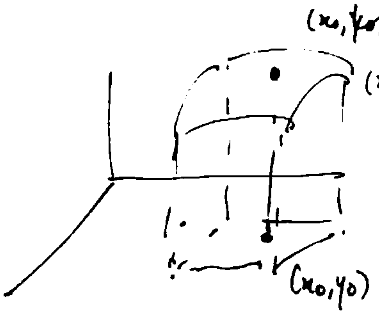
~~① Graph of f~~

⑥ <u>Equation of $T_a(L_c)$.</u>

$$\begin{cases} T_a(L_c) \text{ passes through } a = (x_0, y_0, z_0) \\ \nabla f(a) = \dfrac{\partial f}{\partial x}(a)\vec{i} + \dfrac{\partial f}{\partial y}(a)\vec{j} + \dfrac{\partial f}{\partial z}(a)\vec{k} \perp T_a(L_c) \end{cases}$$

$$\boxed{\frac{\partial f}{\partial x}(a)\left(X - x_0\right) + \frac{\partial f}{\partial y}(a)\left(Y - y_0\right) + \frac{\partial f}{\partial z}(a)\left(Z - z_0\right) = 0.}$$

⑦ ~~Ge~~ <u>Tangent</u> to a graph of the function $h: \mathbb{R}^2 \to \mathbb{R}$.



$$\text{Graph}(h) = \{ (x, y, f(x,y)) \mid (x,y) \in \mathbb{R}^2 \}$$

$$\cancel{z = f(x,y)}$$

$$= \{ (x,y,z) \mid z = f(x,y) \}.$$

Think of it as a level set of a function

$$F: \mathbb{R}^3 \to \mathbb{R}.$$

$$F(x,y,z) = \cancel{z}\ f(x,y) - z.$$

$$\text{Graph}(f) = L_0 \quad (\text{of } F).$$

Equation.

$$\frac{\partial F}{\partial x_0}(a)(X - x_0) + \frac{\partial F}{\partial y_0}(a)(Y - y_0) - \frac{\partial F}{\partial z_0}(a)(Z - z_0) = 0.$$

$$\begin{cases} \frac{\partial F}{\partial x}(a) = \frac{\partial f}{\partial x}(x_0, y_0) \\ \frac{\partial F}{\partial y}(a) = \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial F}{\partial z}(a) = -1. \end{cases}$$

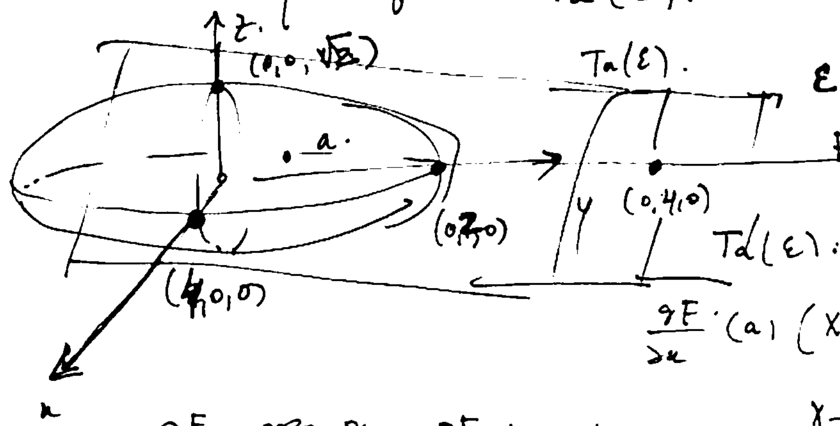
Get $\quad Z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(X - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(Z - z_0).$

the formula that you've been using it before.

(8) Example.

Ellipsoid. $\quad \frac{X^2}{4} + Y^2 + 2Z^2 = 4. \qquad (\mathcal{E})$

Take $a = (2, 1, 1) \in \mathcal{E}$. Find the equation of the tangent plane $T_a(\mathcal{E})$.



$\mathcal{E} = \{F = 4\}$.

$F(x, y, z) = \frac{x^2}{4} + y^2 + 2z^2$

$T_a(\mathcal{E})$:

$$\frac{\partial F}{\partial x}(a)(X - 2) + \frac{\partial F}{\partial y}(Y - 1) + \frac{\partial F}{\partial z}(Z - 1) = 0.$$

$$X - 2 + 2(Y - 1) + 4(Z - 1) = 0$$

$\frac{\partial F}{\partial x} = \frac{x}{2}, \quad \frac{\partial F}{\partial x}(a) = 1$

$\frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial y}(a) = 2, \quad \frac{\partial F}{\partial z} = 4z, \quad \frac{\partial F}{\partial z}(a) = 4.$

$$\boxed{X + 2Y + 4Z - 8 = 0}$$

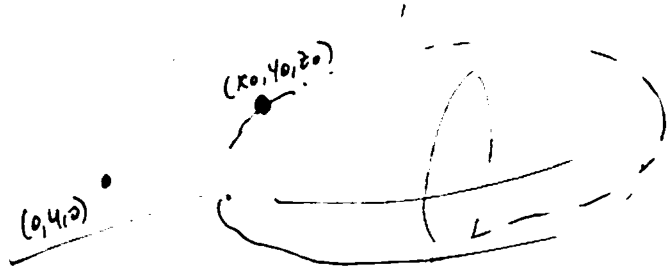
(9) Determine the point where $T_a(\Sigma)$ intersects the y-axis.

Let $x = z = 0$.

$y = 4$.

$T_a(\Sigma) \cap (y\text{-axis}) = \cancel{(0,0,4)}.\ (0,4,0).$

(10). Find all the tangent ~~lines~~ planes to the rugby ball that pass through ~~(0,0,4)~~. $(0,4,0)$



$a = (x_0, y_0, z_0) \in \Sigma.$

$T_a(\Sigma)$: $\frac{x_0}{2}(X - x_0) + 2y_0(Y - y_0) + 4z_0(Z - z_0) = 0.$

$(0,4,0) \in T_a(\Sigma)$: $\frac{x_0}{2}(-x_0) + 2y_0(4 - y_0) + 4z_0(-z_0) = 0.$

$$-\frac{x_0^2}{2} - 2y_0^2 - 4z_0^2 + 8y_0 = 0.$$

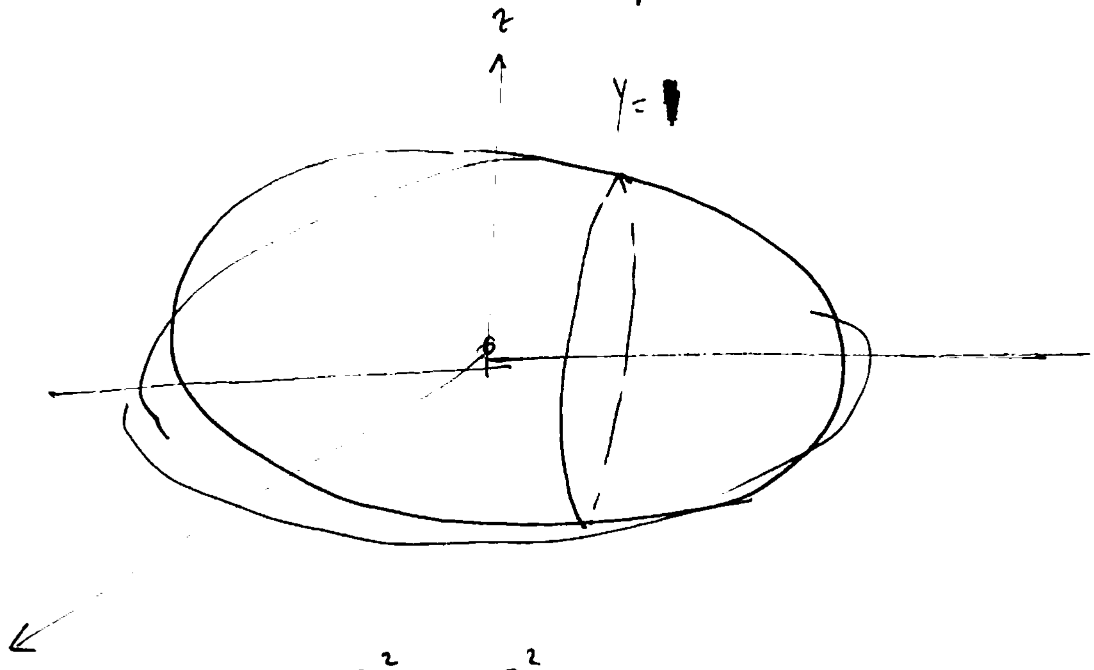
$$\begin{cases} \frac{x_0^2}{4} + y_0^2 + 2z_0^2 - 4y_0 = 0 \\ \frac{x_0^2}{4} + y_0^2 + 2z_0^2 = 4. \end{cases}$$

$4y_0 = 4 \Rightarrow \boxed{y_0 = 1}$

~~$\frac{x_0^2}{4} + y_0^2 + \frac{1}{2} = 4.$~~

~~$x_0^2 + 4y_0^2 = 14.$~~

$\frac{x_0^2}{4} + 1 + 2z_0^2 = 4,$ $\boxed{\frac{x_0^2}{4} + 2z_0^2 = 3.}$



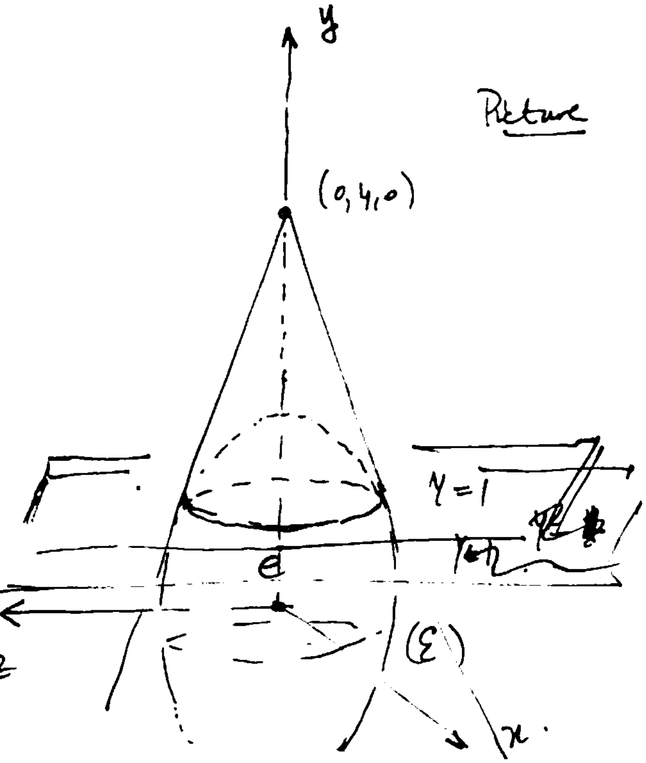
$$\frac{x_0^2}{12} + \frac{2z_0^2}{3} = 1.$$

Sketch

Abc

<u>A</u> : the tangent planes to the points on the ellipse

$$e = \mathcal{E} \cap (y = [illegible]). = \left\{ (x_0, [illegible], z_0) \,\middle|\, \frac{x^2}{4} + z_0^2 = 3 \right\}.$$



<u>Picture</u> The tangent planes envelop around a cone with basis e and vertex (0,4,0).

Leibniz Rule

1. $\frac{d}{dt}\, \underline{c}_1(t)\cdot \underline{c}_2(t) = \underline{c}_1'(t)\cdot \underline{c}_2(t) + \underline{c}_1(t)\cdot \underline{c}_2'(t)$

2. $\frac{d}{dt}\, \underline{c}_1(t)\times \underline{c}_2(t) = \underline{c}_1'(t)\times \underline{c}_2(t) + \underline{c}_1(t)\times \underline{c}_2'(t)$

Applications.

1. $\underline{\gamma}(t)$ a path in $\mathbb{R}^3$ such that $\|\gamma(t)\| =$ constant $= R$, $\forall t$.
Prove that $\underline{\gamma}'(t) \perp \underline{\gamma}(t)$, $\forall t$.

Proof $\quad \frac{d}{dt}\|\gamma(t)\|^2 = 0$,

$$0 = \frac{d}{dt}\, \underline{\gamma}(t)\cdot \gamma(t) = \gamma'(t)\cdot \gamma(t) + \gamma(t)\cdot \gamma'(t)$$

so $\frac{d}{dt}\|\gamma(t)\|^2 = 2\, \underline{\gamma}'(t)\cdot \underline{\gamma}(t)$.

Therefore $\underline{\gamma}'(t)\cdot \underline{\gamma}(t) = 0$

2. Conservation of Energy

Let $V : \mathbb{R}^3 \to \mathbb{R}$ a "potential" function, and $\underline{c}(t)$ the movement of a particle such that

$$\underline{c}''(t) = GM\, \nabla V(\underline{c}(t)).$$

Prove that $\frac{1}{2} m\|c'(t)\|^2 - mV(\underline{c}(t)) =$ constant

Proof $\quad \frac{d}{dt}\left\{\frac{1}{2} m\|c'(t)\|^2 - m\, V(\underline{c}(t))\right\} =$

$$= m\, \underline{c}''(t)\cdot \underline{c}'(t) - m\, \nabla V(\underline{c}(t))\cdot \underline{c}'(t) =$$

$$= m\left\{\underline{c}''(t) - \nabla V(\underline{c}(t))\right\}\cdot \underline{c}'(t) = 0$$

hence $E = \frac{1}{2} m\|c'(t)\|^2 - m\, V(\underline{c}(t)) =$ constant.

Example For $V = \frac{1}{r}$ i.e. $V(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, we have $\nabla V = -\frac{\vec{r}}{r^3}$