Lecture 14 [Wed, Mar 2, 2005]

Mixed Partial Derivatives

$f : \mathbb{R}^3 \to \mathbb{R}$.

Notation. f_{zx}

Assume $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are differentiable and have continuous partial derivatives.

Definition $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

alternative notation

$f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$.

$f_{xz} = (f_x)_z = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial z \partial x}$.

Example 1. Compute second order partial derivatives of

$f(x,y) = x \log y$

Answer $f_x = \log y$ $\quad f_{xx} = 0 \quad f_{xy} = \frac{\partial}{\partial y}(\log y) = \frac{1}{y}$.

$f_y = \frac{x}{y}$ $\quad \begin{cases} f_{yx} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y} \\ f_{yy} = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2}. \end{cases}$

Note $f_{yx} = f_{xy} \to$ "The Order Does not Matter"

Main Theorem If f is C^2 (has continuous, second order partial derivatives), then

$f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$.

Example 2

Compute f_{xyzxz}, for $f(x,y,z) = x\exp\{yz^2\sin(y+z)\}$.

Proof $f_{xyzxz} = f_{yzzxx} = \frac{\partial^3}{\partial z\partial z\partial y}\{f_{xx}\}$.

But $f_x = \exp\{yz^2\sin(y+z)\}$

$f_{xx} = 0$, so $f_{xyzxz} = 0$

Linear Approximations

1) $f: \mathbb{R} \to \mathbb{R}$. diff at x_0

$$\lim_{h\to 0} \boxed{\frac{f(x_0+h) - f(x_0) - h f'(x_0)}{|h|}} = 0$$

$= \omega_1(x_0; h)$

$$f(x_0+h) = f(x_0) + h f'(x_0) + \underbrace{|h|\ \omega_1(x_0;h)}_{\text{"small to first order"}},$$

with $\lim_{h\to 0}\omega_1(x_0;h) = 0$.

Example $f(x_0+\varepsilon) = f(x_0) + \varepsilon f'(x_0) + \underbrace{o(\varepsilon)}_{\text{much smaller than } \varepsilon}$.

Example Compute $\sqrt{4.1}$

$f(x) = \sqrt{x}$. $\quad f'(x) = \frac{1}{2}x^{-1/2}$, $f'(4) = \frac{1}{4}$

$$f(4.1) = f(4+0.1) = f(4) + 0.1\cdot f'(4) + 0.1\ \omega_1(4, 0.1)$$

$$\approx f(4) + 0.1\cdot f'(4)$$

$$= 2 + 0.1\cdot\frac{1}{4} = 2 + 0.025 = 2.025$$

2) $f : \mathbb{R}^n \to \mathbb{R}$, differentiable at x_0. By definition, this means

$$\lim_{h \to 0} \overbrace{\frac{f(x_0+h) - f(x_0) - \nabla f(x_0)\cdot h}{\|h\|}}^{=\, w_1(x_0;\, h)} = 0$$

$$f(x_0+h) = f(x_0) + \nabla f(x_0)\cdot h + \|h\| \underbrace{w_1(x_0; h)}_{\text{small to first order.}}$$

i.e. $f(x_0+h) \cong f(x_0) + \nabla f(x_0)\cdot h$

OR $\boxed{f(x) \cong f(x_0) + \nabla f(x_0)\cdot(x - x_0)}$ ~~first order~~ linear approximation

"x near x_0"

Example.

Compute $\sqrt{(4.01)^2 + (3.98)^2 + (2.03)^2}$.

$f(x,y,z) = x^2 + y^2 + z^2$

$x_0 = (4, 4, 2)$ $\qquad h = (0.01, -0.02, 0.03)$.

$f(x_0+h) \cong f(x_0) + \nabla f(x_0)\cdot h$

$f(x_0) = f(4,4,2) = \sqrt{4^2+4^2+2^2} = 6.$

$\nabla f(x,y,z) = (2x, 2y, 2z)$

$\nabla f(x_0) = (8, 8, 4)$

$\Rightarrow \sqrt{(4.01)^2 + (3.98)^2 + (2.03)^2} \simeq 6 + (8,8,4)\cdot(0.01, -0.02, 0.03)$

$= 6 + -0.08 + 0.12$

$\simeq 6.04$