# MATH 202-MIDTERM 1
# SOLUTIONS

**Problem 1**

**a.** $S$ is the level set $F = 0$, where $F : \mathbb{R}^3 \to \mathbb{R}$ is given by $F(x, y, z) = x^3 - yz - y$. Gradient: $\nabla F(x, y, z) = (3x^2, -z - 1, -y)$. In particular $\nabla F(P) = \nabla F(2, 1, 7) = (12, -8, -1)$. The tangent plane $T_P(S)$ passes through $P(2, 1, 7)$ and has the normal vector $\nabla F(P) = 12i - 8j - k$, hence its equation is

$$12(X - 2) - 8(Y - 1) - (Z - 7) = 0, \quad 12X - 8Y - Z - 9 - 0$$

**b.** Note that the vector $N = 9i - j - 3k$ is normal to the plane $\alpha$. Let $m = (x, y, z)$ a point on $S$ such that $T_m(S)$ is parallel to the plane $\alpha$. We have two conditions on $m$:

$$\begin{cases} F(x, y, z) = 0 & [m \in S] \\ \nabla F(x, y, z) = \lambda N & [T_m(S) \parallel \alpha] \end{cases}$$

Explicitly, this means

$$\begin{cases} 3x^2 = 9\lambda \\ -z - 1 = -\lambda \\ -y = -3\lambda \\ x^3 - yz - y = 0 \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{3\lambda} \\ y = 3\lambda \\ z = \lambda - 1 \\ 3\sqrt{3}\lambda^{3/2} - 3\lambda(\lambda - 1) - 3\lambda = 0 \end{cases}$$

The last equation simplifies to $3\sqrt{3}\lambda^{3/2} = 3\lambda^2 = 0$, i.e. $\lambda \in \{0, 3\}$. The corresponding points on the surface are $(3, 9, 2)$ and $(0, 0, 0)$. [However $\nabla F(0, 0, 0) = 0$ so the tangent plane at $(0, 0, 0)$ is not well-defined, so we're left with $(3, 9, 2)$.]

**Problem 2**

**a.** Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) = \text{dist}^2(\mathbf{x}, \alpha)$. Explicitly $f$ is given by $f(x, y, z) = \frac{(x+2y+z-1)^2}{6}$. Gradient: $\nabla f(x, y, z) = \frac{x+2y+z-1}{3}(i + 2j + k)$.
$\phi(t) = f(\gamma(t))$, hence by the chain rule:

$$\phi'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) = \frac{t^2 + 4t + 2}{3}(1, 2, 1) \cdot (2, 1, 2t) = \frac{1}{3}(t^2 + 4t + 2)(2t + 4)$$

**b.** The distance $d(\gamma(t), \alpha)$ is minimal iff $\phi(t)$ is minimal. To minimize the (one-variable) function $\phi(t)$ one has to look for critical points:

$$\phi'(t) = 0 \Rightarrow t \in \{-2, -2 \pm \sqrt{2}\}$$

If we compare the value of $\phi$ at these points we have

$$\phi(-2 \pm \sqrt{2}) = 0, \quad \phi(-2) = \text{dist}^2((-5, 0, 4), \alpha)\frac{2}{3}$$

Therefore $\phi(t)$ is at a minimum for $t = -2 \pm \sqrt{2}$, which is actually when the path $\gamma(t)$ intersects the plane $\alpha$.

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**Problem 3**

**a.**

$$N = \overrightarrow{BA} \times \overrightarrow{CA} = (2j + k) \times (i + k)$$

$$= \begin{vmatrix} i & j & k \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2i + j - 2k$$

**b.** $P(x, y, z)$ is characterized by the following two properties:

$$\begin{cases} \overrightarrow{OP} \parallel N \\ \overrightarrow{BP} \perp N \end{cases} \Leftrightarrow \begin{cases} \overrightarrow{OP} = \lambda N \\ \overrightarrow{BP} \cdot N = 0 \end{cases}$$

Explicitly

$$\begin{cases} x = 2\lambda \\ y = \lambda \\ z = -2\lambda \\ 2(x-1) + y - 2z = 0 \end{cases}$$

Substitute for $x, y, z$ in the last equation: $2(2\lambda - 1) + \lambda + 4\lambda = 0$, yields $\lambda = \frac{2}{9}$ and $P = (\frac{4}{9}, \frac{2}{9}, -\frac{4}{9})$.

**Problem 4**

**a.** $\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} 0 = 0$. Same way $\frac{\partial f}{\partial y}(0,0) = 0$. In particular $\nabla f(0,0) = 0$.

**b.** $\partial_{\overrightarrow{v}} f(0,0) = \lim_{t \to 0} \frac{f(t\overrightarrow{v}) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(t\overrightarrow{v})}{t}$. But

$$f(t\overrightarrow{v}) = f(t\cos\theta, t\sin\theta) = \frac{t^3(\cos^2\theta \sin\theta + \cos\theta \sin^2\theta)}{t^2(\cos^2\theta + \sin^2\theta)} = t\sin\theta\cos\theta(\sin\theta + \cos\theta)$$

therefore

$$\partial_{\overrightarrow{v}} f(0,0) = \sin\theta\cos\theta(\sin\theta + \cos\theta) = \frac{\sqrt{2}}{2}\sin(2\theta)\sin(\theta + \frac{\pi}{4})$$

**c.** $f$ is **not** differentiable at $(0,0)$ since clearly $\partial_{\overrightarrow{v}} f(0,0) \neq \nabla f(0,0) \cdot \overrightarrow{v}$.

**d.** The (unit) direction of greatest increase is the one for which the directional derivative $\partial_{\overrightarrow{v}} f(0,0)$ is the greatest, that is we need to find $\theta$ which maximizes the expression

$$\frac{\sqrt{2}}{2}\sin(2\theta)\sin(\theta + \frac{\pi}{4})$$

But this is certainly $\leq \frac{\sqrt{2}}{2}$ and it is attained at $\theta = \frac{\pi}{4}$, therefore

$$\overrightarrow{v} = \cos(\frac{\pi}{4})i + \sin(\frac{\pi}{4})j = \frac{\sqrt{2}}{2}(i + j)$$