MIDTER 2, APRIL 6, 2005: SOLUTIONS
MATH 202

1. Problem 1

To find critical points, solve

$$\begin{cases} f_x = 3x^2 - 8x - y = 0 \\ f_y = -x - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 8x - y = 0 \\ y = -x/2 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 8x + \frac{x}{2} = 0 \\ y = -x/2 \end{cases}$$

so $6x^2 - 15x = 3x(2x-5) = 0$, which implies that either $x = 0$ (hence $y = 0$), or $x = 5/2$ (hence $y = -5$).

Thus we get two critical points: $(0,0)$ and $(5/2, -5)$.

For the second derivative test, we need

$$f_{xx} = 6x - 8$$
$$f_{x,y} = -1$$
$$f_{yy} = -2$$

At $(0,0)$ we have

$$H_f(0,0) = \begin{pmatrix} -8 & -1 \\ -1 & -2 \end{pmatrix}$$

which has determinant $= 15$ and since $-8 < 0$ this is a local maximum.

At $(5/2, -5)$

$$H_f(5/2,-5) = \begin{pmatrix} 7 & -1 \\ -1 & -2 \end{pmatrix}$$

and this has determinant -15, indicating a saddle point.

2. Problem 2

Here $h(x,y) = x^3 - y^2 + 1$ and the constraint is $g(x,y) = x^2 + y^2 - 1 \le 0$.

Critical points in the interior: $h_x = 3x^2 = 0$, $h_y = -2y = 0$; hence get $(0,0)$.

Critical points on the boundary: Lagrange multiplier problem

$$\begin{cases} h_x = 3x^2 = 2\lambda x \\ h_y = -2y = 2\lambda y \\ x^2 + y^2 - 1 = 0 \end{cases}$$

If $x = 0$, then $y = \pm 1$. If $y = 0$, then $x = \pm 1$. Otherwise, $x = \frac{2\lambda}{3}$ and $\lambda = -1$, so $x = -\frac{2}{3}$, $y = \pm\frac{\sqrt{5}}{3}$. So we need to compare the values

(x,y)	$(0,0)$	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$	$(-\frac{2}{3}, \frac{\sqrt{5}}{3})$	$(-\frac{2}{3}, -\frac{\sqrt{5}}{3})$
$f(x,y)$	1	2	0	0	0	$\frac{4}{27}$	$\frac{4}{27}$

- $\max f = 2$ is realized at $(1,0)$.
- $\min f = 0$ is realized at three points.

3. Problem 3

Integrating on horizontal slices:

$$\begin{aligned}\int\int_D x\sin(y^3)dxdy &= \int_0^1 \left\{\int_0^{2y} x\sin(y^3)dx\right\} dy \\ &= \int_0^1 2y^2\sin(y^3)dy = -\frac{2}{3}\cos(y^3)\Big|_{y=0}^{y=1} \\ &= \frac{2(1-\cos(1))}{3}\end{aligned}$$

4. Problem 4

a. V is the domain under the graph of $z = 1 + x^2 + y^2$ over the disk $D = \{(x, y)|x^2 + y^2 \leq 1\}$. Therefore:

$$\mathrm{vol}(V) = \int\int_D (1 + x^2 + y^2)dxdy$$

b. Using polar coordinates $[r, \theta]$ on the disk, $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$ and the area element is $dx\,dy = r\,dr\,d\theta$. The integral equals

$$\int_0^{2\pi}\int_0^1 (1+r^2)r\;dr\,d\theta = (\int_0^1 (1+r^2)r\,dr)(\int_0^{2\pi} d\theta) = 2\pi\int_0^1 (r + r^3)dr$$
$$= 2\pi(\frac{1}{2} + \frac{1}{4}) = \frac{3\pi}{2}$$