# DIFFERENTIABILITY IN SEVERAL VARIABLES: SUMMARY OF BASIC CONCEPTS

**1. Partial derivatives** If $f : \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function and $a = (x_0, y_0, z_0) \in \mathbb{R}^3$, then

$$\frac{\partial f}{\partial y}(a) := \lim_{t\to 0} \frac{f(a + t\mathbf{j}) - f(a)}{t} = \lim_{t\to 0} \frac{f(x_0, y_0 + t, z_0) - f(x_0, y_0, z_0)}{t} \tag{1}$$

etc.. provided the limit exists.
Example 1. for $f : \mathbb{R}^2 \to \mathbb{R}$, $\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t}$.
Example 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Then:

- $\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0$
- $\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$

*Note:* away from $(0,0)$, where $f$ is the quotient of differentiable functions (with non-zero denominator) one can apply the usual rules of derivation:

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy(x^2+y^2) - 2x^3 y}{(x^2+y^2)^2}, \quad \text{for } (x,y) \neq (0,0)$$

**2. Directional derivatives.** If $f : \mathbb{R}^2 \to \mathbb{R}$ is a map, $a = (x_0, y_0) \in \mathbb{R}^2$ and $v = \alpha\mathbf{i} + \beta\mathbf{j}$ is a vector in $\mathbb{R}^2$, then by definition

$$\partial_v f(a) := \lim_{t\to 0} \frac{f(a + tv) - f(a)}{t} = \lim_{t\to 0} \frac{f(x_0 + t\alpha, y_0 + t\beta) - f(x_0, y_0)}{t} \tag{2}$$

Example 3. Let $f$ the function from the Example 2 above. Then for $v = \alpha\mathbf{i} + \beta\mathbf{j}$ a unit vector (i.e. $\alpha^2 + \beta^2 = 1$), we have

$$\partial_v f(0,0) = \lim_{t\to 0} \frac{f(tv) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t\alpha, t\beta)}{t} = \alpha^2\beta$$

**3. Definition.** A map $f : \mathbb{R}^2 \to \mathbb{R}$ is *differentiable* at $a \in \mathbb{R}^2$ **if and only if** the following **two** conditions are satisfied:

- $\frac{\partial f}{\partial x}(a)$ and $\frac{\partial f}{\partial y}(a)$ exist; in other words, $\nabla f(a)$ exists.
- $\lim_{\mathbf{x}\to a} \frac{f(\mathbf{x}) - f(a) - \nabla f(a)\cdot(\mathbf{x} - a)}{\|\mathbf{x} - a\|} = 0$

Example 4. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a map such that $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Then $f$ is differentiable at $(0,0)$ iff $\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0)}{\sqrt{x^2+y^2}} = 0$.

**4. Theorem A. If** $f : \mathbb{R}^2 \to \mathbb{R}$ is **differentiable** at $a$ and $v$ is a vector in $\mathbb{R}^2$, **then**:

$$\partial_v f(a) = \nabla f(a) \cdot v \qquad \text{[dot product]} \tag{3}$$

Example 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ the function from Example 2 again. We saw that $\nabla f(0,0) = 0$, while $\partial_v f(0,0) = \alpha^2\beta$ for a unit vector $v = \alpha\mathbf{i} + \beta\mathbf{j}$. Hence clearly $\partial_v f(0,0) \neq \nabla f(0,0)\cdot v$, at least say, for $v = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$. In view of Theorem A, we conclude that $f$ is not differentiable at $(0,0)$.

One can also rely on the definition of differentiability: suppose $f$ is differentiable and see if we can obtain a contradiction. If this was the case, we would have $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$, which is not true as we could check for $x = y = t$, $t \to 0$.

**5. Theorem B.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ a map and $a \in \mathbb{R}^2$.
**If**:

- $f$ has partial derivatives (in a neighborhood of $a$) AND
- the partial derivatives $f_x, f_y$ are continuous at $a$,

**then** $f$ is differentiable.

In other words :

$$\mathbf{C^1} \Rightarrow \text{Differentiable} \tag{4}$$

yet the converse is **not** true.

Example 6. The function $f$ from Example 2 satisfies

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{2xy^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

One can check that $\frac{\partial f}{\partial x}$ is **not** continuous at $(0,0)$ (by taking $x = y = t \to 0$), but this is not the reason why $f$ is **not** differentiable at $(0,0)$, but rather a consequence of it. To illustrate this last point, consider

Example 7. Let $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then $h$ is differentiable everywhere, including at $x = 0$ where we have

$$h'(0) = \lim_{t\to 0} \frac{h(t) - h(0)}{t} = \lim_{t \to 0} t \sin(1/t) = 0$$

yet $h'$ is not continuous at 0, since

$$h'(x) = \begin{cases} 2x\sin(1/x) - \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

In other words, $h$ is differentiable without being $\mathbf{C^1}$.

**6. Chain rule.** If $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ are differentiable then $h = g \circ f$ is differentiable and for $a \in \mathbb{R}^m$ we have and $h'(a) = g'(b)f'(a)$, in other words

$$D_h(a) = D_g(b)D_f(a)$$

where $b = f(a) \in \mathbb{R}^n$, $D_h(a)$ is the matrix of partial derivatives of $h$ at $a$, etc...

*Important application.* If $\gamma(t) : \mathbb{R} \to \mathbb{R}^3$ is a differentiable map (path) and $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$ is differentiable as well, then $\phi = f \circ \gamma$ is differentiable as a function $\varphi : \mathbb{R} \to \mathbb{R}$, and its derivative is given by $\varphi'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t)$.

**7. Geometric meaning of gradient.** If $F : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $S$ is the level set $S = \{F = c\} = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = c\}$, then for a point $a \in S$, then

$$\nabla F(a) \perp T_a(S) \tag{5}$$

Example 8. Consider a differentiable function $h : R^2 \to \mathbb{R}$. Then graph($h$) (the graph of $h$) seen as a surface in $\mathbb{R}^3$ is simply the level set $\{F = 0\}$ of the function of 3 variables $F(x,y,z) = f(x,y) - z$. Then, for $a = (x_0, y_0) \in \mathbb{R}^2$ and $m = (x_0, y_0, z_0)$ the corresponding point on the graph ($z_0 = h(x_0, y_0)$), the vector

$$\nabla F(x_0, y_0, z_0) = \frac{\partial h}{\partial x}(x_0, y_0)\mathbf{i} + \frac{\partial h}{\partial y}(x_0, y_0)\mathbf{j} - \mathbf{k}$$

is normal to $T_m(\text{graph}(h))$. This allows us to find the equation of the tangent plane to the graph.