PRACTICE EXAM

Note: the format of the final exam might change (almost certainly). This practice exam is intended to give you an idea of the level of difficulty of the exam (the actual exam will probably be shorter).

1. Part I

1.1. Let T the (interior of) the triangle with vertices $A(1,1,1)$, $B(3,1,0)$ and $C(2,4,6)$. (T is a surface in $\mathbb{R}^3$.) Determine the unit vector $\overrightarrow{n}$ with positive $\mathbf{k}$ component which is normal to the plane of triangle T.

1.2. Describe the boundary ∂T of T and specify the orientation compatible with the normal vector n. (You might want to draw a picture to make you answer as clear as possible.)

1.3. Compute the area of T by any method you want.

1.4. Compute the integral $\int_{\partial T} \overrightarrow{F}$, where $\overrightarrow{F} = -y\mathbf{i} + x\mathbf{j}$ and ∂T has the orientation from part 1.2).

2. Part II

In this part the temperature on the spherical surface $S : x^2 + y^2 + z^2 = 4$ is given by $T(x, y, z) = xy + yz$. Also, we consider the vector field $\overrightarrow{F} = \nabla T$.

2.1. Determine all the hot spots on S.

2.2. Determine a vector field of the form $\overrightarrow{H} = f(x, y, z)\mathbf{j}$ such that $\nabla \times \overrightarrow{H} = y\mathbf{i}$.

2.3. Determine a vector field $\overrightarrow{G}$ such that $\operatorname{curl} \overrightarrow{G} = \overrightarrow{F}$. Why does such a vector exist? Verify your answer.
(Hint: you might want to write your vector field as sum of four vector fields and use the previous part.)

2.4. Determine the flux of $\overrightarrow{F}$ through the part of S given by $y \geq \sqrt{2}$. Use your favorite method.

3. Part III

For this part, solve four out the following five questions.

3.1. Compute the volume of the ball of radius R by any method you want.

3.2. Let Σ be the surface parametrized by

$$\Phi(u, v) = (u,\ u^3 \cos v,\ u^3 \sin v)$$

with $(u, v) \in [0, 1] \times [0, 2\pi]$. Calculate the normal vector $\overrightarrow{n}_\Sigma$ and the area of the resulting surface.

3.3. Use Green's theorem to express the area of the domain D in $\mathbb{R}^2$ bounded by the parabola $y = x^2$ and the line $y = x$.

3.4. Let $D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq y \leq 1\}$. Let $f : \mathbb{R} \to \mathbb{R}$ an arbitrary one-variable continuous function. Prove the following identity:

$$\int\int_D f(x)f(y)\, dxdy = \frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

3.5. Let C a closed path in $\mathbb{R}^2$, not passing through the origin. Assume C is without self-intersections and oriented counterclockwise. Consider the vector field

$$\overrightarrow{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

Prove that

$$\frac{1}{2\pi}\int_C \overrightarrow{F} = \begin{cases} 1, & \text{if } (0,0) \text{ is in the interior of } C \\ 0, & \text{otherwise} \end{cases}$$