PROBLEM ON GREEN'S THEOREM

0.1. Consider the vector field $\overrightarrow{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$.

a) Compute $\int_{C_R} \overrightarrow{F}$ where C_R is the circle of radius R centered at the origin in the xy-plane (oriented counterclockwise).

b) Prove that $\nabla \times \overrightarrow{F} = 0$.

c) Let γ a simple (that is, without self-intersections) closed curve in $\mathbb{R}^2$, not passing through the origin. Use Green's formula to prove that

$$\frac{1}{2\pi}\int_\gamma \overrightarrow{F} = \begin{cases} 1, & \text{if } (0,0) \text{ is in the interior of } \gamma \\ 0, & \text{otherwise} \end{cases}$$

Hint: if the curve contains $(0,0)$ in its interior, choose a small radius R such that the circle C_R is also in the interior of γ. Then apply Green's theorem on the domain $D = \text{Interior}(\gamma) - \text{Interior}(C_R)$.

d) Let γ an arbitrary closed curve in $\mathbb{R}^2$, not passing through the origin. Prove that $\frac{1}{2\pi}\int_\gamma \overrightarrow{F} = m$, where m is the number of times the curve γ winds around $(0,0)$ counterclockwise.