PRACTICE PROBLEMS

1. Vector geometry

1.1. Find the equation of the plane that passes through $A(1, 2, 0)$, $B(0, 1, -2)$, $C(4, 0, 1)$.

1.2. Find the distance from the point $(2, 1, -2)$ to the plane

$$x - 2y + 2z + 5 = 0$$

1.3. Given two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, do the equations

$$\overrightarrow{v} \times \overrightarrow{a} = \overrightarrow{b} \quad \text{and} \quad \overrightarrow{v} \cdot \overrightarrow{a} = \|a\|$$

determine the vector $\overrightarrow{v}$ uniquely? If so, find an explicit formula of $\overrightarrow{v}$ in terms of $\overrightarrow{a}$ and $\overrightarrow{b}$.

2. Tangent planes & lines

2.1. Find the points on the surface $z = x^2y^2 + y + 1$ where the tangent plane (to the surface) is parallel to the plane $-2x - 3y + z = 1$.

2.2. Find the tangent plane to the graph of the function

$$f(x, y) = \frac{x}{x + y}$$

at the point $(x_0, y_0) = (1, 0)$.

2.3. Find a unit vector normal to the graph of $f(x, y) = e^x y$ at the point $(-1, 1)$.

2.4. Show that the graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ are tangent at $(0, 0)$.

2.5. Consider the surface $z^2 = x^2 + y^2$. Is the tangent plane at the point $(0, 0, 0)$ well defined?

3. Curves in $\mathbb{R}^3$

3.1. The curve $\mathbf{c}(t) = (t, t^2, t^3)$ crosses the plane $4x+2y+z = 24$ at a single point. Find that point and calculate the cosine of the angle between the tangent vector at $\mathbf{c}$ at that point and the normal vector to the plane.

3.2. A particle travels on the surface of a fixed sphere of radius R centered at the origin, i.e.

$$\|\gamma(t)\| = R, \quad \forall t$$

where $\gamma(t) \in \mathbb{R}^3$ is the position of the particle at time t. Prove that the velocity is always perpendicular on the position vector, i.e.

$$\gamma'(t) \cdot \overrightarrow{\gamma(t)} = 0, \quad \forall t$$

3.3. **a)** Let $V : \mathbb{R}^3 \to \mathbb{R}$ given by $V(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, or in other words $V(\overrightarrow{r}) = \frac{1}{r}$. Compute ∇V.

b) The equation of motion of a planet that orbits the Sun satisfies the equation of motion

$$c''(t) = -GM\frac{\overrightarrow{c(t)}}{\|c(t)\|^3}$$

where $c(t) = (x(t), y(t), z(t))$ is the position of the planet at time t. Prove that the vector (angular momentum)

$$\overrightarrow{L} = \overrightarrow{c(t)} \times c'(t)$$

is independent of time.

c) Prove that the quantity (energy)

$$E = \frac{1}{2}m\|c'(t)\|^2 - \frac{GMm}{\|c(t)\|}$$

is independent of time

3.4. The position of a particle in time is given by

$$c(t) = (\cos(\pi t), \sin(\pi t), \pi t)$$

At time $t_0 = \frac{5\pi}{2}$ the particle is freed of any constraints and starts travelling along the tangent at the constant speed $v = v(t_0)$. Determine how long does it take (starting from t_0) the particle to hit the wall given by the equation $x = 0$.

4. Limits

4.1. Determine whether the following limit exists:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x) - 2x + y}{x^3 + y}$$

5. Differentiability

5.1. Compute f_x, f_y, f_z and evaluate them at the indicated points

$$\textbf{a)} \quad f(x,y,z) = \sqrt{x^2+y^2+z^2}; \quad (3,0,4)$$

$$\textbf{b)} \quad f(x,y,z) = x\sqrt{x^2+y^2+z^2}; \quad (2,1,3)$$

$$\textbf{c)} \quad f(x,y,z) = xy + e^x \cos y; \quad (1, \frac{\pi}{2}, 0)$$

5.2. Compute the matrix of partial derivatives of the function $f(x,y) = (x+y, x-y, xy)$.

5.3. **a)** True or false: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(1,0)$ exist, then f is differentiable at $(1,0)$.

b) True or false: if $\partial_{\overrightarrow{v}} f$ is the directional derivative of f along the vector $\overrightarrow{v}$, then $\partial_{\overrightarrow{j}} f(1,0,1) = f_y(1,0,1)$.

c) True or false: if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(0,1)$ exist, then f is continuous at $(1,0)$.

d) Give the definition of the directional derivative $\partial_{\overrightarrow{v}} f(x_0, y_0)$ where $\overrightarrow{v}$ is some unit vector in $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ a function.

e) Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, give the definition of the tangent plane at $(1, 0)$ to the graph of f (provided it exists).

f) Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is *differentiable* and $\frac{\partial f}{\partial x}(1, 0)$ and $\frac{\partial f}{\partial y}(1, 0)$ exist.
True or false: if $\overrightarrow{v} = p\,\overrightarrow{i} + q\,\overrightarrow{j}$ is a unit vector, then $\partial_{\overrightarrow{v}} f(1, 0) = p$.

g) Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is *continuous* and $\frac{\partial f}{\partial x}(1, 0)$ and $\frac{\partial f}{\partial y}(1, 0)$ exist.
True or false: if $\overrightarrow{v} = p\,\overrightarrow{i} + q\,\overrightarrow{j}$ is a unit vector, then $\partial_{\overrightarrow{v}} f(1, 0) = \nabla f(1, 0) \cdot \overrightarrow{v}$.

5.4. **a)** Argue that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y, z) = (x + e^z + y, yx^2)$ is differentiable.

b) Compute an approximate value for $f(1.1, 0, -0.9)$.

5.5. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = (0, 0) \end{cases}$.

a) Compute $\frac{\partial f}{\partial x}(0, 0)$, $\frac{\partial f}{\partial y}(0, 0)$, and $\partial_{\overrightarrow{v}} f(0, 0)$ for any unit vector $\overline{v}$.

b) Is f differentiable at $(0, 0)$?

6. Chain rule

6.1. Use the chain rule to find u_x, u_y, u_z for $u = e^x \cos(yz^2)$.

6.2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ a differentiable map, such that $g_z(0, 1) = 1$, $g_y(0, 1) = 2$. Determine the rate of change of g (at (1,0)) along the circle centered at the origin and radius 1. In other words, compute $\frac{\partial g}{\partial \theta}$ at $(1, 0)$ where $x = r \cos \theta$ and $y = r \sin \theta$.

6.3. Let $(x(t), y(t))$ a path in the plane $0 \le t \le 1$, and let $f(x, y)$ a C^1 function of two variables. Assume that

$$x'(t) f_x(x(t), y(t)) + y'(t) f_y(x(t), y(t)) \le 0$$

Prove that $f(x(1), y(1)) \le f(x(0), y(0))$.

6.4. Let $f : \mathbb{R} \to \mathbb{R}$ a differentiable function and $h = f(\frac{x+y}{x-y})$. Prove that $h(z, y)$ satisfies the equation

$$x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = 0$$

6.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + y$ and $h : \mathbb{R}^2 \to \mathbb{R}$, $h(u) = (\sin(3u), \cos(8u))$. Let $g(u) = f(h(u))$, for $u \in \mathbb{R}$. Compute $g'(0)$ both directly *and* by using the chain rule.

7. Gradients

7.1. Let $f(x, y, z) = (\sin(xy))e^{-z^2}$. In what direction from $(1, \pi, 0)$ should one proceed to increase f most rapidly? Express your answer as a unit vector.