PRACTICE PROBLEMS-ANSWERS TO SOME PROBLEMS

1. Vector geometry

1.1. Given two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, do the equations

$$\overrightarrow{v} \times \overrightarrow{a} = \overrightarrow{b} \quad \text{and} \quad \overrightarrow{v} \cdot \overrightarrow{a} = \|a\|$$

determine the vector $\overrightarrow{v}$ uniquely? If so, find an explicit formula of $\overrightarrow{v}$ in terms of $\overrightarrow{a}$ and $\overrightarrow{b}$.

Answer. The answer is yes. Clearly if a and b are not orthogonal then there is no solution. So assume a b are orthogonal vectors. Let θ the angle between v and a.

$$v \times a = b \Rightarrow \|b\| = \|a\|\|v\|\ |\sin\theta|$$
$$v \cdot a = \|a\| \Rightarrow \|a\| = \|a\|\|v\|\ \cos\theta$$

Hence

$$\|v\|^2(\cos^2\theta + \sin^2\theta) = 1 + \frac{\|b\|^2}{\|a\|^2}$$

so that the length of v is determines

$$\|v\| = \sqrt{1 + \frac{\|b\|^2}{\|a\|^2}}$$

also $\theta = \pm\arccos(1/\|v\|)$. Therefore v is the vector in the perpendicular to b, of given length, such that the angle between x and b is $\pm\frac{\|a\|}{\sqrt{\|a\|^2+\|b\|^2}}$, depending weather x, a, b is *oriented* or not.

A more elegant solution: we may assume $a \perp b$, otherwise there is no solution. Then the vectors $a, b, a \times b$ form an orthogonal *basis*, hence one can write v as a linear combination

$$v = xa + yb + za \times b, \quad x, y, z \text{ real numbers}$$

We will then take inner products of v with the vectors $a, b, a \times b$.
First, $v \cdot a = x\|a\|^2$. But $v \cdot a = \|a\|$, hence $x = \frac{1}{\|a\|}$.
Also $v \cdot b = y\|b\|^2$. But $v \times a = b$ implies $v \perp b$, hence $y = 0$.
Finally $v \cdot (a \times b) = z\|a \times b\|^2$. But $v \cdot (a \times b) = b \cdot (v \times a)$ (think of the 3×3 determinant expressing the cross product) $= b \cdot b = \|b\|^2$. Therefore $z = \frac{\|b\|^2}{\|a \times b\|^2}$, hence

$$x = \frac{1}{\|a\|}\, a + \frac{\|b\|^2}{\|a \times b\|^2}\, a \times b$$

2. Tangent planes & lines

2.1. Find the points on the surface $z = x^2y^2 + y + 1$ where the tangent plane (to the surface) is parallel to the plane $\alpha : -2x - 3y + z = 1$.

Answer: two planes $AX + BY + CZ + D = 0$ and $A'X + B'Y + C'Z + D = 0$ are parallel (or perpendicular) iff their corresponding normal vectors $N = Ai+Bj+Ck$ and $N' = A'i + B'j + C'k$ are parallel (perpendicular).
Our surface is the level set $S = \{F = -1\}$, where $F(x, y, z) = x^2y^2 + y - z$. Let $m(x, y, z) \in S$. The normal vector to the tangent plane $T_m(S)$ is $N = \nabla F(m)$. The normal vector to the plane α is $N' = -2i - 3j + k$. Then $T_m(S) \parallel \alpha$ iff there exists $\lambda \in \mathbb{R}$ such that $\nabla F(m) = \lambda N'$, in other words we need to solve the system

$$\begin{cases} \frac{\partial F}{\partial x}(x, y, z) = -2\lambda \\ \frac{\partial F}{\partial y}(x, y, z) = -3\lambda \\ \frac{\partial F}{\partial z}(x, y, z) = \lambda \\ F(x, y, z) = -1 \end{cases}$$

in other words

$$\begin{cases} 2xy^2 = -2\lambda \\ 2x^2y + 1 = -3\lambda \\ -1 = \lambda \\ x^2y^2 + y - z = -1 \end{cases}$$

hence $\lambda = -1$, $x^2y = x^2y = 1$, $x^2y^2 + y - z = -1$, i.e. $x = y = 1$, $z = 3$, so $m = (1, 1, 3)$.

2.2. Find a unit vector normal to the graph of $f(x, y) = e^x y$ at the point $(-1, 1)$.

Answer:

$$\mathrm{Graph}(f) = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\} = \{F = 0\} \quad \text{[level set]}$$

where $F(x, y, z) = f(x, y) - z$. Let $m = (-1, 1, f(-1, 1)) = (-1, 1, e^{-1})$ the corresponding point on the graph. Then $\nabla F(m)$ is a normal vector to the tangent plane $T_m(\{F = 0\})$. But

$$\nabla F(m) = (\frac{\partial f}{\partial x}(-1, 1), \frac{\partial f}{\partial y}(-1, 1), -1) = \dots$$

2.3. Show that the graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ are tangent at $(0, 0)$.

Answer: $f(0, 0) = g(0, 0) = 0$, so there graphs share the point $O(0, 0, 0)$. One has to check then that $graph(f)$ and $graph(g)$ have the same tangent plane at $(0, 0)$. To see this, one has to check that $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial g}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0) = \frac{\partial g}{\partial y}(0, 0)$.

2.4. Consider the surface $z^2 = x^2 + y^2$. Is the tangent plane at the point $(0, 0, 0)$ well defined?

Answer: not really since this is the level set $\{F = 0\}$ where $F(x, y, z) = x^2 + y^2 - z^2$ and $\nabla F(0, 0, 0) = 0$.

3. Curves in $\mathbb{R}^3$

3.1. The curve $\mathbf{c}(t) = (t, t^2, t^3)$ crosses the plane $4x + 2y + z = 24$ at a single point. Find that point and calculate the cosine of the angle between the tangent vector at $\mathbf{c}$ at that point and the normal vector to the plane.

Answer: first one needs to find t_0 such that $c(t_0) \in$ plane, i.e. solve the equation $4t + 2t^2 + t^3 = 24$. Then one needs to find (cosine of) the angle between (the velocity at t_0) $c'(t_0)$ and the normal to the plane $N = 4i + 2j + k$. For this use the inner product.

3.2. A particle travels on the surface of a fixed sphere of radius R centered at the origin, i.e.

$$\|\gamma(t)\| = R, \quad \forall t$$

where $\gamma(t) \in \mathbb{R}^3$ is the position of the particle at time t. Prove that the velocity is always perpendicular on the position vector, i.e.

$$\gamma'(t) \cdot \overrightarrow{\gamma(t)} = 0, \quad \forall t$$

Answer: let $\phi(t) = \|\gamma(t)\|^2 = \gamma(t)\dot{\gamma}(t)$. Then $\phi(t) = R^2$, for all t, hence $\phi'(t) = 0$. On the other hand by the Leibniz rule, $\phi'(t) = 2\gamma'(t) \cdot \gamma(t)$, hence $\gamma'(t) \cdot \gamma(t) = 0$.

3.3. **a)** Let $V : \mathbb{R}^3 \to \mathbb{R}$ given by $V(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, or in other words $V(\overrightarrow{r}) = \frac{1}{r}$. Compute ∇V.

b) The equation of motion of a planet that orbits the Sun satisfies the equation of motion

$$c''(t) = -GM \frac{\overrightarrow{c(t)}}{\|c(t)\|^3}$$

where $c(t) = (x(t), y(t), z(t))$ is the position of the planet at time t. Prove that the vector (angular momentum)

$$\overrightarrow{L} = \overrightarrow{c(t)} \times c'(t)$$

is independent of time.

c) Prove that the quantity (energy)

$$E = \frac{1}{2} m \|c'(t)\|^2 - \frac{GMm}{\|c(t)\|}$$

is independent of time

Answer: differentiate the angular momentum with respect to time

$$\frac{d}{dt} L(t) = c'(t) \times c'(t) + c(t) \times c''(t) = c(t) \times c''(t) = c(t) \times \left(-\frac{GM}{\|c(t)\|^3} c(t)\right) = 0$$

since $v \times v = 0$, for any vector v.

Differentiate the energy with respect to t:

$$\begin{aligned}\frac{d}{dt} E &= \frac{d}{dt}\left(\frac{m}{2} c'(t) \cdot c'(t) - GMV(c(t))\right) = mc'(t) \cdot c''(t) - GM \nabla V(c(t)) \cdot c'(t) \\ &= mc'(t) \cdot \left(c''(t) - GM \frac{c(t)}{\|c(t)\|^3}\right) = 0\end{aligned}$$

by the equation of motion.
Note: since the vector L is constant in time, since $\overrightarrow{c(t)}$ is perpendicular on L (by definition) it follows that $c(t)$ moves in a fixed plane, i.e. the orbit is planar.

3.4. The position of a particle in time is given by

$$c(t) = (\cos(\pi t), \sin(\pi t), \pi t)$$

At time $t_0 = \frac{9}{4}$ the particle is freed of any constraints and starts travelling along the tangent at the constant speed $v = v(t_0)$. Determine how long does it take (starting from t_0) the particle to hit the wall given by the equation $x = 0$.

Answer: the tangent is given by the velocity vector $c'(t_0)$. It means that at time $t > t_0$, the position of the particle is given by

$$l(t) = l(t_0) + (t - t_0)c'(t_0) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{9\pi}{4}) + (t - \frac{9}{4})(-\frac{\pi\sqrt{2}}{2}, \frac{\pi\sqrt{2}}{2}, \pi)$$

We need to determine t such that the points $l(t)$ is in the plane $x = 0$, i.e. need to solve the equation

$$\frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{2}(t - \frac{9}{4}) = 0$$

so $t = \frac{1}{\pi} + \frac{9}{4}$.

4. Limits

4.1. Determine whether the following limit exists:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x) - 2x + y}{x^3 + y}$$

Answer: $\lim_{t\to 0} f(0,t) = 0$ trivially, while $\lim_{t\to 0} f(t,0) = \lim_{t\to 0} \frac{\sin(2t)-2t}{t^3} = \frac{4}{3}$ by L'Hopital. So the answer is no.

5. Differentiability

5.1. **a)** True or false: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(1,0)$ exist, then f is differentiable at $(1,0)$. [false]

b) True or false: if $\partial_{\overrightarrow{v}} f$ is the directional derivative of f along the vector $\overrightarrow{v}$, then $\partial_{\overrightarrow{j}} f(1,0,1) = f_y(1,0,1)$. [true]

c) True or false: if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(0,1)$ exist, then f is continuous at $(1,0)$ [false]

g) Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is *continuous* and $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(1,0)$ exist.
True or false: if $\overrightarrow{v} = p\,\overrightarrow{i} + q\,\overrightarrow{j}$ is a unit vector, then $\partial_{\overrightarrow{v}} f(1,0) = \nabla f(1,0) \cdot \overrightarrow{v}$. [false- you need f to be differentiable, and mere existence of partial derivatives does not ensure this]

5.2. **a)** Argue that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y, z) = (x+e^z+y, yx^2)$ is differentiable.

Answer: the partial derivatives of its components

$$(f_1)_x = 1, (f_1)_y = 1, (f_1)_z = e^z, (f_2)_x = 2xy, (f_2)_y = x^2, (f_2)_z = 0$$

are all continuous (everywhere) functions $(f_1)_x, \cdots : \mathbb{R}^3 \to \mathbb{R}$, therefore f is differentiable everywhere.

6. Chain rule

6.1. Use the chain rule to find u_x, u_y, u_z for $u = e^x \cos(yz^2)$.

Answer: $u_y = -e^x \sin(yz^2) \cdot (2z^2) - 2e^x z^2 \sin(yz^2)$.

6.2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ a differentiable map, such that $g_z(0, 1) = 1$, $g_y(0, 1) = 2$. Determine the rate of change of g (at (1,0)) along the circle centered at the origin and radius 1. In other words, compute $\frac{\partial g}{\partial \theta}$ at $(1, 0)$ where $x = r \cos\theta$ and $y = r \sin\theta$.

Answer: for general x, y we have

$$\begin{aligned}\frac{\partial g}{\partial \theta} &= \frac{\partial g}{\partial x}(r\cos\theta, r\sin\theta)(-r\sin\theta) + \frac{\partial g}{\partial y}(r\cos\theta, r\sin\theta)(r\cos\theta)\\ &= -r\sin\theta g_x(x, y) + r\cos\theta g_y(x, y), \quad x = r\cos\theta, y = r\sin\theta\end{aligned}$$

Therefore $\frac{\partial g}{\partial \theta}(1, 0) = \frac{\partial g}{\partial y}(1, 0)$.

6.3. Let $(x(t), y(t))$ a path in the plane $0 \le t \le 1$, and let $f(x, y)$ a C^1 function of two variables. Assume that

$$x'(t)f_x(x(t), y(t)) + y'(t)f_y(x(t), y(t)) \le 0$$

Prove that $f(x(1), y(1)) \le f(x(0), y(0))$.

Answer: let $\phi(t) = f(x(t), y(t))$. The condition in the problem reads (via chain rule) $\phi'(t) \le 0$, therefore $\phi(t)$ is decreasing, hence $f(x(0), y(0)) = \phi(0) \ge \phi(1) = f(x(1), y(1))$.

7. Gradients

7.1. Let $f(x, y, z) = (\sin(xy))e^{-z^2}$. In what direction from $(1, \pi, 0)$ should one proceed to increase f most rapidly? Express your answer as a unit vector.

Answer: since f is differentiable (argue this point), the direction of greatest increase is that of the gradient, so

$$v = \frac{\nabla f(1, \pi, 0)}{\|\nabla f(1, \pi, 0)\|} = \ldots$$